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To
Professor Cajori
H. E. Slaught

SOLID GEOMETRY

WITH

PROBLEMS AND APPLICATIONS

BY

H. E. SLAUGHT, Ph.D.

ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CHICAGO

AND

N. J. LENNES, Ph.D.

INSTRUCTOR IN MATHEMATICS IN COLUMBIA UNIVERSITY

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PREFACE.

The important features by which the Solid Geometry seeks to accomplish the two main purposes stated in the preface of the Plane Geometry are:

1. *The concepts of three-dimensional space are made clear by many simple illustrations and exercises.* At best the average pupil comes but slowly and gradually to a full comprehension of space forms and relations. Pages 4, 5, 8, 10, 11, 17, 18, etc., will show how the pupil is aided in understanding and appreciating these forms and relations at the outset. In this connection, Chapter VI on Graphic Representation is of fundamental importance, since it exhibits to the eye the functional relations among varying geometric forms.

2. *Interesting concrete applications are interspersed throughout the text.* It is the aim profoundly to impress upon the mind of the pupil the practical significance of certain fundamental theorems in solid geometry. For instance, how many pupils in the ordinary study of the theorems on the ratios of surfaces and volumes of similar figures realize that there is in them any connection with the possibility of successfully launching a steamship a mile long? See the exercises on pages 112, 113, 166, 167. For illustrations of other interesting and useful applications, see pages 59, 66, 67, 156, 157.

3. *The logical structure is made more complete and more prominent than in the Plane Geometry.* Solid Geometry is studied by more mature pupils who have been led by gradual stages in the Plane Geometry to a knowledge and appre-

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ciation of deductive reasoning. Hence the axioms are stated and applied in strictly scientific form and at the precise points where they are to be used. For instance, see §§ 5, 7, 9, 83, 105, 107, 138, 157, etc.

Note also the consistent and scientific definitions of all solids, not as bounded *portions of space*, but as *configurations in space*, the uniform conception in all higher mathematical usage. See §§ 68, 70, 99, 132, 149, 200.

4. *Throughout both parts of this Geometry a consistent scheme has been followed in the presentation of incommensurables and the theory of limits.* In Chapters I to VI of the Plane Geometry the idea of "approach" is made clear by many concrete illustrations, and the theorems involving this idea are shown to hold for *all possible approximations.* In the final chapter of the Plane Geometry rigorous proofs of these theorems are given and in far simpler terminology than is found in current text-books. In the Solid Geometry this latter method is followed throughout Chapters I to VI, thus giving a complete and scientific treatment up to that point. In Chapter VII of the Solid Geometry the theory of limits is presented in such a way as to leave nothing to be unlearned or compromised in later mathematical work. This chapter may be omitted without affecting the logical completeness of the book.

H. E. SLAUGHT.
N. J. LENNES.

CHICAGO AND NEW YORK,
March, 1911.

CONTENTS.

SOLID GEOMETRY.

CHAPTER I.

LINES AND PLANES IN SPACE.

1. A figure in plane geometry is restricted so that all its points lie in the same plane. Such figures are called **two-dimensional figures.**

A straight line is a *one-dimensional figure*, and a point has *no dimensions.*

2. **A three-dimensional figure** is a combination of points, lines, and surfaces not all parts of which lie in the same plane.

E.g. the six surfaces of the walls, floor, and ceiling of the schoolroom form a three-dimensional figure. This figure *is not the room itself.* The room is the space inclosed by the figure.

3. **Solid geometry** treats of the properties of three-dimensional figures.

DETERMINATION OF A PLANE.

4. Since we are to consider points and lines not all lying in the same plane, it is of first importance to be able to distinguish one plane from another.

This is all the more important, since three-dimensional figures have to be represented by pencil or crayon drawings on the plane of the paper or blackboard. Models of paper, wire, etc., may be constructed by the pupils.

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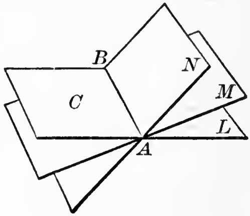
5. **Axiom I.** *If two points of a straight line lie in a plane, the whole line lies in the plane.*

Since a line is endless, it follows from this axiom that a plane is unlimited in all its directions.

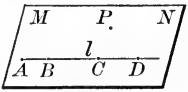
6. While two points determine a line, this is not sufficient in the case of a plane.

E.g. suppose a plane L contains the points A and B which determine the line AB. If the plane L be revolved about the line AB as an axis, it may occupy indefinitely many positions, as L, M, N, but there is only one position in which it contains a third given point C outside of the line AB.



7. **Axiom II.** *Through three points not all in the same straight line one and only one plane can be passed.*

8. THEOREM. *A plane is determined by* (1) *a line and a point not in that line,* (2) *two intersecting lines,* (3) *two parallel lines.*



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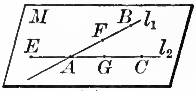

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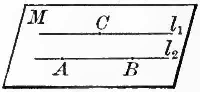


(3)

Proof: (1) **Let l be the given line and P the point outside.**

Take any two points A and B on l. Then A, B, and P determine a plane, which we call M (Ax. II).

Take any other two points C and D on l. Then C, D, and P determine a plane, which we call N (Ax. II).

By Ax. I, N contains the points A and B. Hence N is the plane determined by A, B, and P, since it contains these points. That is, N is the same plane as M.

Therefore one and only one plane is determined by a line and a point not on that line.

(2) **Let l_1 and l_2 be two intersecting lines.**

Take A the intersection of l_1 and l_2, and B and C any other points, one on l_1 and the other on l_2.

Then A, B, and C determine a plane M in which both l_1 and l_2 lie, since A and B lie on l_1 and A and C on l_2 (Ax. I).

Now M is the only plane in which both l_1 and l_2 lie, for any other three points E, F, G, on l_1 and l_2 (not all in the same line) determine the same plane M, since they all lie in it.

Hence, two intersecting lines determine a plane.

(3) **Let l_1 and l_2 be two parallel lines.**

By definition l_1 and l_2 lie in a plane M.

Now M is the only plane in which both l_1 and l_2 lie, for any three points A, B, C, on l_1 and l_2 (not all on the same line) determine this plane M, since they all lie in it.

Hence, two parallel lines determine a plane.

9. Axiom III. *If two planes have a point in common, then they have at least another point in common.*

10. THEOREM. *Two intersecting planes have a straight line in common, and no points in common outside of this line.*

Proof: If two planes intersect, they must have at least two points in common (Ax. III).

But these two points determine a straight line which lies wholly in each plane (Ax. I).

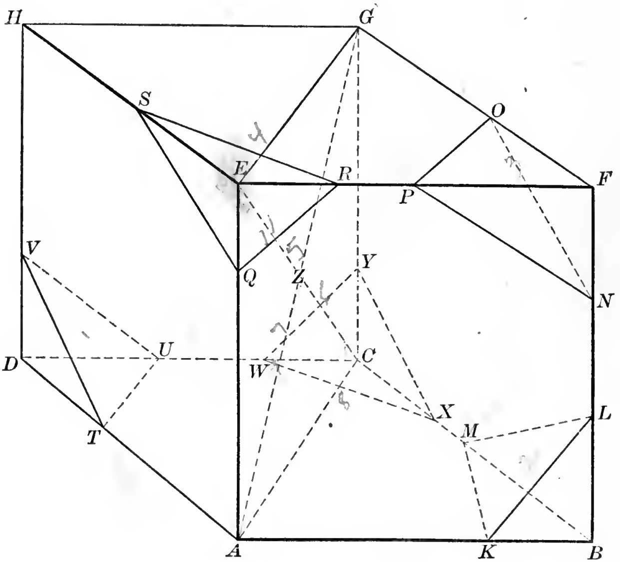
Hence, there is a straight line common to the two planes.

Now prove that these two planes can have no point in common outside of this line unless the planes are identical.

11. EXERCISES.

The diagram on this page represents a three-dimensional figure in the shape of an ordinary box. In this figure the points A, K, B do not determine a plâne, since they all lie on the same straight line, while A, C, E do not lie in a straight line, and hence determine a plane.

1. In this figure pick out twelve sets of three points each which do not determine planes, and also twelve sets which do determine planes.



2. Describe the relation to the three-dimensional figure of each plane determined in Ex. 1. Thus, the plane WXY cuts off the lower right-hand corner at the back.

3. Read each of the six bounding planes in this figure, using various different sets of points or lines. Thus, the face $ABFE$ is determined by the points B, L, K, or by A and the line PN, or by lines AB and EF.

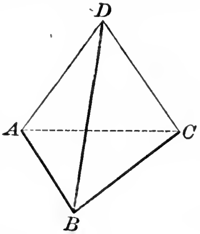
4. Describe the position of each of the following planes in the figure: AEC, ACF, BCE, HCA, ADG, BDF, HEC.

5. Pick out six planes, each determined by two parallel lines in the figure.

6. Decide whether each of the six planes in Ex. 5 may also be determined by two intersecting lines of the figure; by a line and a point outside of it.

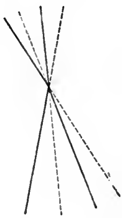
7. Using the point Z, pick out six planes, each determined by it and two other points of the figure, as AZD, AZB, etc.

8. Using the schoolroom or a room at home, imagine all the planes constructed which are involved in the foregoing questions. Do the same, using a small box or paper model.



9. Does a stool with three legs always stand firmly on a flat floor? One with four legs? Give reasons. On what condition does a stool with four legs stand firmly on a flat floor?

10. In the figure the point D is not supposed to lie in the same plane as ABC. Hence, does C lie in the plane of ABD? Does B lie in the plane of ACD?



11. How many planes are determined by four points which do not all lie in the same plane?

12. How many planes are determined by four lines which all meet in a point, but no three of which lie in the same plane?

13. What is the locus of all points common to two intersecting planes?

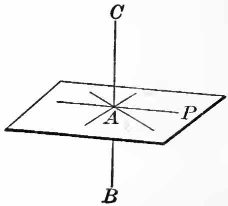
12. The demonstrations and constructions of solid geometry consist largely in applying the theorems already known in plane geometry to figures lying in various planes determined by points and lines in space, as illustrated in the preceding exercises.

The *next* following problems are typical in this respect.

THEOREMS ON PERPENDICULARS.

13. Definitions. A line is said to be **perpendicular to a plane** if it is perpendicular to every line of the plane passing through the point in which it meets the plane. In this case the plane is also **perpendicular to the line.** The point in which the perpendicular meets the plane is the **foot of the perpendicular.**

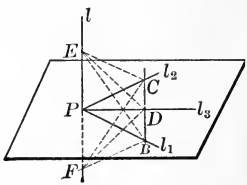
NOTE. It is obvious that at a point in a line there may be *many lines in space* which are perpendicular to it. For instance, if AP is $\perp BC$, rotate AP about BC as an axis, keeping $\angle PAC = \angle PAB$. Then AP remains $\perp BC$ in every one of its positions.



Thus, all spokes of a wheel are perpendicular to the axle.

14. A line or plane is said to be **constructed** whenever the points which determine it are constructed.

15. PROBLEM. *Through a given point to construct a plane perpendicular to a given line.*



Given: the line *l* and the point *P*.

To construct a plane through P perpendicular to l.

Construction. (*a*) Let P be on the given line l. Through P construct two lines l_1 and l_2, each perpendicular to l. Then the plane determined by l_1 and l_2 is the required plane.

Proof: Connect B and C, any two points different from P on l_1 and l_2 respectively.

Through P draw any line l_3 meeting BC in D.

On l lay off $PE = PF$ and complete the figure as shown.

Now prove (1) $\triangle EBC \cong \triangle FBC$, (2) $\angle EBD = \angle FBD$, (3) $\triangle EBD \cong \triangle FBD$, (4) $\triangle EPD \cong \triangle FPD$, (5) $\angle EPD = \angle FPD$. Hence, $PD \perp l$.

This proof holds for every such line l_3 except the line through P parallel to BC. In this case we can select another point C' on l_2 so that l_3 shall not be parallel to BC'.

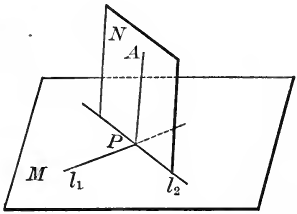
Since PD is any line through P in the plane of l_1 and l_2, it follows by definition that the plane determined by l_1 and l_2 is perpendicular to l.

(*b*) If the point P is not on the line l, draw a line from P perpendicular to l at some point Q, and then as above construct a plane through Q perpendicular to l.

16. EXERCISE.

Prove that through a point there is not more than one plane perpendicular to a given line.

17. Problem. *At a point in a plane to construct a line perpendicular to the plane.*



Given: the point P in the plane M.

To construct a line perpendicular to M at P.

Construction. Let l_1 be any line in M through P.

Pass a plane N through $P \perp l_1$ (§ 15).

Let plane N cut plane M in l_2 (§ 10).

In plane N erect $PA \perp l_2$.

Then PA is the perpendicular required.

Proof: Show that PA is $\perp$ to both l_1 and l_2 and hence to their plane, that is, to the plane M.

18. **EXERCISES.**

1. At a point in a plane only one line can be erected perpendicular to the plane.

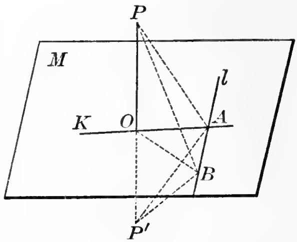
PROOF. Suppose a second perpendicular is possible. Then we shall have two lines as l_3 and l_4 perpendicular to M at P. Let the plane of l_3 and l_4 intersect M in a line l_5. Then l_3 and l_4 are both $\perp$ to l_5 and lie in the same plane with it, which is impossible.

2. Take two pieces of cardboard in one of which a line is drawn at random, and in the other a line is drawn perpendicular to an edge. Place them so as to illustrate the construction in § 17.

19. PROBLEM. *From a point outside a plane to construct a line perpendicular to the plane.*

Given: the plane M and the point P outside it.

To construct a line from $P \perp M$.



Construction. Let l be any line in plane M.

From P draw $PA \perp l$, and in plane M draw $AK \perp l$ at A.

Finally, draw $PO \perp AK$. Then PO is the required line perpendicular to M.

Proof: Through O in M draw any other line OB meeting l.

Prolong PO to P', making $OP' = OP$, and connect points as shown in the figure.

Then line $l \perp$ plane PAP' (why?).

Now show that (1) $PA = P'A$, (2) $\triangle APB \cong \triangle AP'B$, (3) $\triangle BOP \cong \triangle BOP'$.

Hence, $PO \perp M$. (Why?)

20 **EXERCISES.**

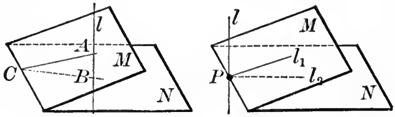
1. Only one perpendicular to a plane can be drawn from a point outside the plane.

SUGGESTION. Show that the hypothesis of two perpendiculars from an outside point to a plane leads to a contradiction.

2. A perpendicular is the shortest distance from a point to a plane.

3. A line cannot be perpendicular to each of two intersecting planes.

SUGGESTIONS. (1) If a line l is perpendicular to the planes M and N at the points A and B, and C is a point in their intersection, then $\triangle ABC$ would contain two right angles. (2) If l is perpendicular to M and N at a point P in their intersection, pass a plane through l, meeting M and N in l_1 and l_2.



4. A line which is perpendicular to each of two lines at their point of intersection is perpendicular to their plane. See § 15.

5. All perpendiculars to a line at a point in it lie in the same plane, namely, the plane determined by any two of them.

The following are theorems proved in the preceding constructions and exercises.

21. THEOREM. *Through any given point there is one and only one plane perpendicular to a given line.*

22. THEOREM. *Through any given point there is one and only one line perpendicular to a given plane.*

23. THEOREM. *All lines perpendicular to a line at a point lie in one and the same plane.*

24. THEOREM. *A line perpendicular to each of two intersecting lines is perpendicular to their plane.*

25. THEOREM. *The shortest distance from a point to a plane is the perpendicular to the plane.*

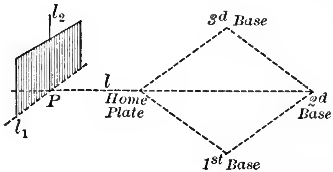
26. EXERCISES.

1. Show how a carpenter could use the theorem of § 24 to stand a post perpendicular to the floor, by use of two ordinary steel squares.

2. If a steel square be made to rotate about one of its edges as an axis, what kind of surface does the other edge describe?

3. In making long timbers perpendicular to each other, a carpenter may use simply a measuring rod, such as a ten-foot pole. Show how to make a timber perpendicular to the floor by this process.

4. Show how a back-stop on a ball field can be adjusted perpendicular to the line through second base and the home plate. What theorems of solid geometry are used?



5. If a plane is perpendicular to a line-segment PP' at its middle point, prove: (1) Every point in the plane is equally distant from P and P'; (2) every point equally distant from P and P' lies in this plane. What is the locus of all points in space equidistant from P and P'? See § 127, Plane Geometry.

6. Given the points A and B not in a plane M. Find the locus of all points in M equidistant from A and B.

SUGGESTION. All such points must lie in the plane M and also in the plane which is the perpendicular bisector of the segment AB.

Is there any position of the points A and B for which there is no such locus? For which the locus contains the whole plane M?

7. On a line find a point equidistant from two given points not on the line.

Is there any position of the points for which there is no such point on the line? For which all points on the line satisfy this condition?

8. Find the locus of all points in space equidistant from the three vertices of a triangle.

9. Find the locus of all points in a given plane equidistant from the vertices of a triangle not in the plane.

Is there any position of the triangle for which this locus contains more than one point? No point?

10. Ask and answer questions similar to the two preceding, using the vertices of a square instead of a triangle; the vertices of a rectangle; of any polygon which can be inscribed in a circle.

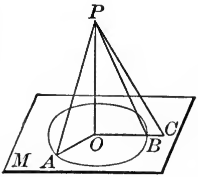
11. Find the locus of all points in space equidistant from the points of a circle and also of all points in a plane equidistant from the points of a circle not in that plane. Discuss as in Ex. 9.

12. Find the locus of all points equidistant from two given points A and B, and also equidistant from two points C and D. Discuss.

13. In geometry of two dimensions, how many lines may be perpendicular to a given line at a given point? In geometry of three dimensions?

14. In how many points can a straight line cut a plane? In how many points may it cut a *curved surface*, such as a stovepipe? In how many points can a straight line cut a circle? In how many points can a plane cut a circle, the plane being distinct from the plane of the circle?

15. State and prove a theorem of solid geometry corresponding to the theorem in the Plane Geometry, comparing the lengths of oblique segments cutting off equal distances from the foot of a perpendicular.

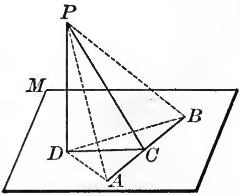


Also state and prove the converse.

State and prove a theorem comparing the lengths of segments cutting off unequal distances. See §§ 112, 115, 116, Plane Geometry.

16. Find the locus of all points in a plane which are equally distant from a given point outside the plane. If a perpendicular be drawn to the plane from this point, how is its foot related to this locus?

17. If in the figure $PD \perp$ plane M, and $DC \perp AB$ any line of the plane, prove that $PC \perp AB$.



SUGGESTION. Lay off $CA = CB$, and compare triangles.

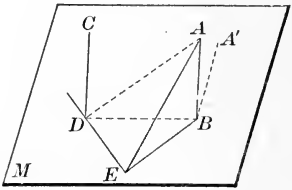
18. If in the same figure $PD \perp M$, and $PC \perp AB$ any line of the plane, prove that $DC \perp AB$.

THEOREMS ON PARALLELS.

27. THEOREM. *If two lines are perpendicular to the same plane, they are parallel.*

CONVERSELY. *If one of two parallel lines is perpendicular to a plane, the other is also.*

Given: (1) AB and CD each $\perp$ to the plane M.



To prove that $AB \parallel CD$.

Proof: Draw BD and make $DE \perp DB$.

Take points A and E so that $BA = DE$, and draw AD, AE, and BE.

Now prove (1) $\triangle ABD \cong \triangle BDE$. (2) $\triangle ADE \cong \triangle ABE$.

Hence, $\angle ADE$ is a right angle, and DC, DA, and DB all lie in the same plane. (Why?)

Therefore CD and AB are in the same plane and perpendicular to the same line BD. (Why?)

Hence, $AB \parallel CD$.

Given: (2) $AB \parallel CD$ and $CD \perp M$.

To prove that $AB \perp M$.

Proof: If AB is not $\perp M$, suppose $A'B \perp M$. Then $A'B \parallel CD$. But through B there is only one line $\parallel CD$.

Hence, $A'B$ and AB coincide.

But $A'B$ was taken $\perp M$. Hence, $AB \perp M$.

HISTORICAL NOTE. The above proof for the direct case is the one given by Euclid. Many proofs have been given, but this seems the most elegant.

28. COROLLARY. *Two lines in space, each parallel to the same line, are parallel to each other.*

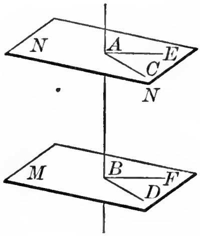
SUGGESTIONS. Let $l_1 \parallel l_3$ and $l_2 \parallel l_3$. To prove $l_1 \parallel l_2$. Take a plane $M \perp l_3$ and finish the proof.

29. Definitions. Two planes which do not meet are said to be **parallel.**

A straight line and a plane which do not meet are said to be **parallel.**

30. THEOREM. *If each of two planes is perpendicular to the same line, they are parallel.*

CONVERSELY. *If one of two parallel planes is perpendicular to a line, the other is also.*



Given: (1) $M \perp AB$ and $N \perp AB$.

To prove that $M \parallel N$.

Proof: Suppose M and N to meet in some point P, and show that this leads to a contradiction.

Given: (2) $M \parallel N$ and $M \perp AB$.

To prove that $N \perp AB$.

Proof: Through AB pass a plane cutting M in BD and N in AC, and also a second plane cutting M in BF and N in AE.

Then $BD \parallel AC$, for if they could meet, then M and N would meet, which is contrary to the hypothesis.

Likewise $BF \parallel AE$.

Complete the proof, showing that $AB \perp AC$, and $AB \perp AE$, and hence $AB \perp N$.

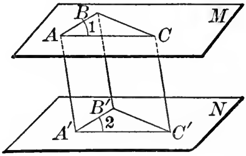
31. COROLLARY. *If a plane intersects two parallel planes, the lines of intersection are parallel.*

32. THEOREM. *If a straight line is parallel to a given plane, it is parallel to the intersection of any plane through it with the given plane.*

SUGGESTION. If l_1 is the given line, and l_2 the intersection of the plane through l_1 with the given plane, show that l_1 and l_2 lie in the same plane and cannot meet.

33. COROLLARY. *If a line outside a plane is parallel to some line in the plane, then the first line is parallel to the plane.*

34. THEOREM. *If two intersecting lines in one plane are parallel, respectively, to two intersecting lines in another plane, then the two planes are parallel, and the corresponding angles formed by the lines are equal.*



Given: the planes M and N in which $AB \parallel A'B'$, and $AC \parallel A'C'$.

To prove that $M \parallel N$ and $\angle 1 = \angle 2$.

Outline of proof: (1) *To prove* $M \parallel N$.

If M and N could meet, let l be their line of intersection. Then neither AB nor AC can meet l since each is $\parallel N$ by § 33. Hence $AB \parallel l$ and $AC \parallel l$, which is impossible. (Why?)

(2) *To prove* $\angle 1 = \angle 2$, study the following analysis:

Lay off $AB = A'B'$, $AC = A'C'$, and draw BC, $B'C'$, AA', BB', and CC'.

Then $\angle 1 = \angle 2$ if $\triangle ABC \cong \triangle A'B'C'$, which is true if $BC = B'C'$. But $BC = B'C'$ if $BCC'B'$ is a ▱, which is so if $BB' = CC'$ and $BB' \parallel CC'$. This last is true if $ABB'A'$

2 93

and $ACC'A'$ are ▱s, for then $BB' = CC' = AA'$ and $BB' \parallel AA' \parallel CC'$.

Now reverse the order of these steps and give the proof in full, with all the reasons.

35. COROLLARY. *If two angles in space have their sides respectively parallel, the angles are either equal or supplementary.*

State this in full detail as in §§ 106–108, Plane Geometry, and show how it applies to the above figure.

36. **EXERCISES.**

1. Show that parallel line-segments included between parallel planes are equal, and hence that parallel planes are everywhere equally distant.

2. Find the locus of all points at a given distance from a given plane; also of all points equally distant from two parallel planes.

3. Show how to determine a plane parallel to a given plane and at a given distance from it. How would you place three shelves parallel to each other and a foot apart?

4. Show that two straight lines in space may not meet and yet not be parallel.

5. Show that through any given line a plane may be passed parallel to any other given line in space.

SUGGESTION. If l_1 and l_2 are the given lines, through any point in l_1, draw $l_3 \parallel l_2$, and show that the plane determined by l_1 and l_3 is the required plane $\parallel l_2$.

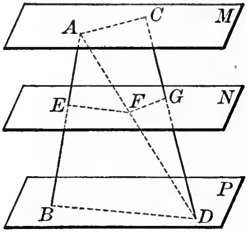
6. Through a given point pass a plane parallel to each of two given lines in space.

SUGGESTION. Through the given point pass lines $\parallel$ to each of the given lines, then use § 33.

7. Show that through a point outside a plane any number of lines can be drawn parallel to the plane. How are all these parallels situated?

8. Find the locus of all lines through a fixed point parallel to a fixed plane.

37. THEOREM. *If two straight lines are cut by three parallel planes, the intercepted segments on one line are in the same ratio as the corresponding segments on the other.*



Given: the lines AB and CD cut by the planes M, N, and P,

To prove that $\frac{AE}{EB} = \frac{CG}{GD}$.

Proof: Connect the points A and D, and let the plane determined by AD and CD cut the planes M and N in AC and FG respectively. And let the plane of AB and AD cut N and P in EF and BD respectively.

Now show that $AC \parallel FG$ and $EF \parallel BD$, and hence complete the proof, giving all the reasons.

38. COROLLARY. *Parallel planes which intercept equal segments on any transversal line intercept equal segments on every transversal line.*

39. **EXERCISES.**

1. Show how to find all the points on the floor of the schoolroom which are equally distant from one of the lower corners of the window sill and one of the upper corners of the opposite door.

2. If the spaces between four shelves are 5, 8, and 10 inches respectively, and a slanting rod intersecting them has a 7-inch segment between the first two shelves, find the other two segments of the rod.

3. If we attempt to stand up a ten-foot pole in a room eight feet high, find the locus of the foot of the pole on the floor when the top is kept at a fixed point on the ceiling.

4. Show that, if three line-segments not in the same plane are equal and parallel, the triangles formed by joining their extremities, as in the figure of § 34, are congruent, and their planes are parallel.

5. Given a plane M and a point P not in M. Find the locus of the middle points of all segments connecting P with points in M.

6. Given a plane M and a point P not in M. Find the locus of a point which divides in a given ratio each segment connecting P with a point in M: (*a*) if the segments are divided internally; (*b*) if they are divided externally.

7. The locus required in Ex. 6 consists of two planes, each parallel to the given plane M. Are these two planes equally distant from M?

8. Show that a plane containing one only of two parallel lines is parallel to the other.

9. If in two intersecting planes a line of one is parallel to a line of the other, then each of these lines is parallel to the line of intersection of the planes.

10. Show that three lines which are not concurrent must all lie in the same plane, if each intersects the other two.

11. Show that three planes, each of which intersects the other two, have a point in common unless their three lines of intersection are parallel.

Suggestion. Suppose two of the intersection lines are not parallel, and meet in some point O. Then show that the other line of intersection passes through O, and hence that O is the point common to all three planes.

12. Given two intersecting planes M and N. Find the locus of all points in M at a given distance from N.

13. Given two non-intersecting lines l_1 and l_2. Find the locus of all lines meeting l_1 and parallel to l_2.

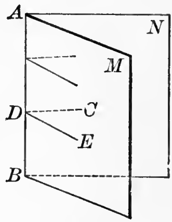
14. Prove that the middle points of the sides of any quadrilateral in space are the vertices of a parallelogram.

Suggestion. Use the fact that a line bisecting two sides of a triangle is parallel to the third side. Note that the four vertices of a quadrilateral in space do not necessarily all lie in the same plane.

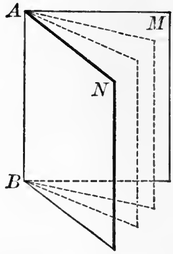
DIHEDRAL ANGLES.

40. Definitions. The part of a plane on one side of a line in it is called a **half-plane.** The line is called the **edge** of the half-plane. Two half-planes meeting in a common edge form a **dihedral angle.** The common edge is the **edge** of the angle and the half-planes are its **faces.**

Lines in the faces of a dihedral angle perpendicular to its edge at a common point form a plane angle, which is called **the plane angle of the dihedral angle.**



Thus, in the figure, the half-planes M and N have the common edge AB, and form the dihedral angle $M–AB–N$, read by naming the two faces and the edge. The $\angle CDE$, whose sides are $CD \perp AB$ in N and $ED \perp AB$ in M is the plane angle of the dihedral angle $M–AB–N$.

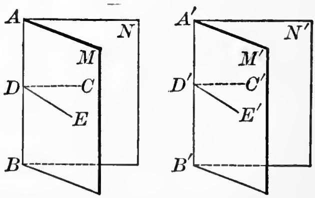


41. By § 34 all plane angles of a dihedral angle are equal to each other.

A dihedral angle may be thought of as **generated** by the rotation of a half-plane about its edge. The magnitude of the angle depends solely upon the *amount of rotation.*

42. Two dihedral angles are equal when they can be so placed that their faces coincide.

43. THEOREM. *Two dihedral angles are equal if their plane angles are equal.*



Given: the dihedral angles $M–AB–N$ and $M'–A'B'–N'$ in which the plane $\angle$s CDE and $C'D'E'$ are equal.

To prove that

$M–AB–N = M'–A'B'–N'$.

Proof: Place the equal $\angle$s CDE and $C'D'E'$ in coincidence.

Then the edges AB and $A'B'$ must also coincide, since they are both perpendicular to the plane CDE at the point D (§ 22).

Face M then coincides with M', since its determining lines, AB and DE, coincide respectively with those of M', namely, $A'B'$ and $D'E'$.

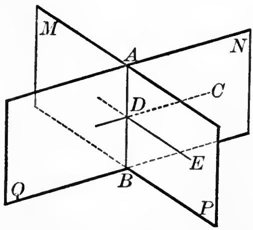
Likewise, face N coincides with N'.

Hence, $M–AB–N = M'–A'B'–C'$ since their faces have been made to coincide (§ 42).

44. THEOREM. *State and prove the converse of the preceding theorem.*

45. Definitions. It follows from §§ 43 and 44 that the plane angle of a dihedral angle may be regarded as its **measure.**

A dihedral angle is **right, acute,** or **obtuse** according as its plane angle is right, acute, or obtuse.



Two dihedral angles are **adjacent, vertical, supplementary,** or **complementary,** according as their corresponding plane angles, with a common vertex, are adjacent, vertical, supplementary, or complementary.

In the figure pick out all the dihedral angles and describe them and their relations, (1) if $\angle CDE$ is acute, (2) if $\angle CDE$ is a right angle.

46. EXERCISES.

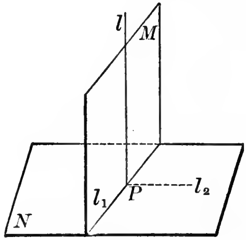
1. State theorems on dihedral angles corresponding to those on plane angles in §§ 64, 68, 69, 72–76, Plane Geometry.

2. State theorems concerning two planes cut by a transversal plane corresponding to those on lines in §§ 90, 92, 93, 97, 99, Plane Geometry.

Note that in Exs. 1 and 2 the proofs are exactly analogous to those in the Plane Geometry.

47. Definition. Two planes are **perpendicular to each other** if their dihedral angle is a right angle.

48. THEOREM. *If a line is perpendicular to a plane, every plane containing this line is perpendicular to the plane.*



Given: a line $l \perp$ plane N at P,

To prove that a plane M containing l is $\perp N$.

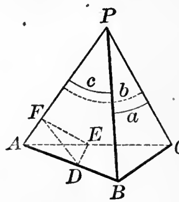
Proof: Let l_1 be the intersection of N and M. In N draw $l_2 \perp l_1$ at P.

Then $\qquad l \perp l_1, l_2 \perp l_1$, and $l_2 \perp l$. $\qquad$ (Why?)

Hence, $\qquad M \perp N$. $\qquad$ (Why?)

49. **EXERCISES.**

1. Show that the above theorem is equivalent to the following: If a plane is perpendicular to a line lying in another plane, then the first plane is perpendicular to the second.



2. Name all the dihedral angles in the accompanying figure.

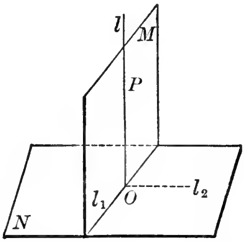
3. Find the locus of all points equidistant from two given parallel planes and also equidistant from two given points.

4. Find the locus of all points at a given distance from a given plane and equidistant from two given points.

Discuss Exs. 3 and 4 for the various cases possible.

5. How can the theorem of § 48 be used to erect a plane perpendicular to a given plane?

50. THEOREM. *If two planes* M *and* N *are mutually perpendicular, and if* P *is any point in* M, *then a line through* P *perpendicular to* N *lies wholly in* M *and is perpendicular to the line of intersection of* M *and* N.



Given: plane $M \perp$ plane N, and P any point in M. Let l_1 be the intersection of M and N and let l be a line through $P \perp N$.

To prove that l lies wholly in M and that $l \perp l_1$.

Proof: (1) **Let P be any point in M outside of l_1.**

Suppose l does not lie wholly in M.

Then through P draw a line l' in the plane M perpendicular to l_1 at O. Through O draw line l_2 in $N \perp l_1$. Then l_2 and l' form a right angle (§ 47). Hence $l' \perp N$ (§ 24).

But from P there can be but one perpendicular to N.

Hence, l coincides with l', and we have l lying wholly in M and $l \perp l_1$.

(2) **Let the point P be in the intersection l_1.**

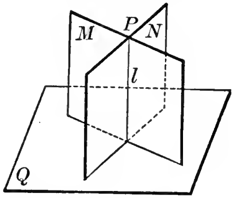
The proof is similar to case (1). Give it in full.

51. EXERCISES.

1. Show that if in the figure of § 50, $M \perp N$ and l is a line in M such that $l \perp l_1$, then $l \perp N$.

2. Through a given point, on or outside of a given plane, how many planes can be constructed perpendicular to the given plane?

52. THEOREM. *If a plane is perpendicular to each of two planes, it is perpendicular to their line of intersection.*



Given: $M \perp Q$ **and** $N \perp Q$ **and** l **the intersection of** M **and** N.

To prove that $Q \perp l$.

Proof: From a point P common to M and N draw a line $l' \perp Q$.

Then l' lies wholly in both M and N. (Why?)

Hence, l' and l are the same line.

That is, $l \perp Q$, or $Q \perp l$.

53. EXERCISES.

1. Any plane ⊥ to the edge of a dihedral angle is ⊥ to each of its faces.

2. If three lines are ⊥ to each other at a common point, then each is ⊥ to the plane of the other two.

3. Through a given line in space pass a plane ⊥ to a given plane. How many such planes can be constructed?

SUGGESTION. From some point in the given line draw a line ⊥ to the given plane. Then use § 48.

4. What is the answer to the question in Ex. 3 in case the given line is perpendicular to the given plane?

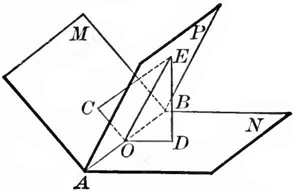
5. Through a given point in space pass a plane ⊥ to each of two given planes. How many such can be passed?

SUGGESTION. Consider the relation of the required plane to the intersection of the two given planes.

6. What is the answer to the question in Ex. 5 in case the given point is on the line of intersection of the given planes?

54. Definition. A plane through the edge of a dihedral angle **bisects the angle** if it forms equal angles with the faces.

55. THEOREM. *The locus of all points in space equally distant from the faces of a dihedral angle is the half-plane bisecting the angle.*



Given: the plane *P* bisecting the dihedral ∠ *M–AB–N*.

To prove: (1) that any point E in P is equally distant from M and N, and (2) that any point which is equally distant from M and N lies in P.

Outline of proof: (1) Draw $EC \perp M$ and $ED \perp N$. Then the plane CED cuts M, N, and P in OC, OD, and OE respectively and is $\perp$ to both M and N. (Why?)

Then $\angle EOD$ is the measure of $P–AB–N$ and $\angle EOC$ is the measure of $P–AB–M$. (Why?)

Now use $\triangle$ EOC and EOD to show that $EC = ED$.

(2) Take E any point such that $EC = ED$, and let P be the plane through AB containing E. Now argue as in § 125, Plane Geometry, to show that $\angle EOD = \angle EOC$, and hence that P' is the same plane as the bisector plane P.

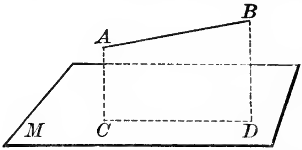
Give the proof in full.

56. COROLLARY. *If from any point within a dihedral angle perpendiculars are drawn to the faces, the angle between the perpendiculars is the supplement of the dihedral angle.*

57. Definition. The **projection of a point** on a plane is the foot of the perpendicular from the point to the plane.

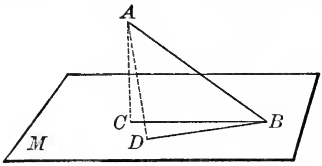
The **projection of any figure** on a plane is the locus of the projections of all points of the figure on the plane.

58. THEOREM. *The projection of any straight line on a plane is a straight line in that plane.*



Outline of proof: Pass a plane through the given line and ⊥ to the given plane. Is this always possible? (Why?) Now show that the intersection of these two planes contains the projection of every point of the given line upon the given plane.

59. THEOREM. *The acute angle formed by a straight line with its own projection on a plane is the least angle which it makes with any line in that plane.*



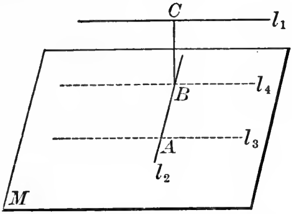
Outline of Proof: Let BC be the projection of AB, and let BD be any other line in M through B.

Lay off $BD = BC$ and draw AD.

In the △ ABD and ABC show that $\angle ABD > \angle ABC$.

60. Definition. The angle between a plane and a line oblique to it is understood to mean the acute angle between the line and its projection upon the plane.

61. PROBLEM. *To construct a common perpendicular to two non-parallel lines in space.*



Given: l_1 and l_2, two non-parallel lines, and also non-intersecting.

To construct a line BC perpendicular to each of them.

Construction: Through A, any point in l_2, draw $l_3 \parallel l_1$.

Let M be the plane determined by l_2 and l_3.

Through l_1 pass a plane $P \perp M$ and meeting M in l_4.

At B the intersection of l_2 and l_4, in the plane P, erect $BC \perp M$.

Then BC is the required perpendicular.

Proof: Give the proof in full.

62. EXERCISES.

1. If a line is $\perp$ to a plane, show that its projection is a point.

2. If a line-segment is $\parallel$ to a plane, show that its projection is a segment equal to the given segment.

3. If a line-segment is oblique to a plane, show that its projection is less than the given segment.

4. If a line-segment 6 in. long makes an angle of 30° with a plane, find the length of its projection on the plane. If it makes an angle of 45°; an angle of 60°.

5. If two parallel lines meet a plane, they make equal angles with it. (Why?) Is the converse true?

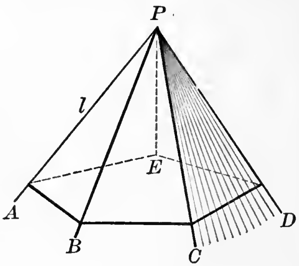
6. If a line cuts two parallel planes, it makes equal angles with them. (Why?) Is the converse true?

7. If two parallel line-segments are oblique to a plane, their projections on the plane are in the same ratio as the given segments.

POLYHEDRAL ANGLES.

63. Definitions. Given a convex polygon and a point P not in its plane. If a half-line l with end point fixed at P moves so that it always touches the polygon and is made to traverse it completely, it is said to generate a **convex polyhedral angle.**

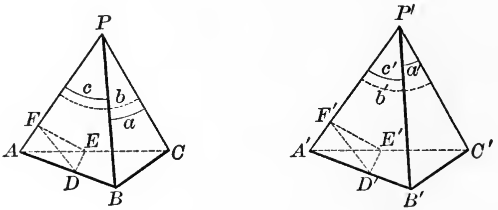
The fixed point is the **vertex** of the polyhedral angle, and the rays through the vertices of the polygon are the **edges** of the polyhedral angle.



Any two consecutive edges determine a plane, and the portion of such a plane included between its edges is called a **face** of the polyhedral angle.

The plane angles at the vertex are called the **face angles** of the polyhedral angle. A polyhedral angle having three faces is called a **trihedral angle.**

64. THEOREM. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, the dihedral angles opposite the equal face angles are equal.*



Given: the trihedral angles P and P', in which $\angle a = \angle a'$, $\angle b = \angle b'$, $\angle c = \angle c'$.

To prove that the pairs of dihedral angles are equal whose edges are PA and $P'A'$, PB and $P'B'$, PC and $P'C'$.

Proof: Let P and P' be cut by the planes ABC and $A'B'C'$, making $PA = PB = PC = P'A' = P'B' = P'C'$.

On the edges PA and $P'A'$ lay off $AF = A'F'$, and through F and F' pass planes $\perp$ to PA and $P'A'$ respectively.

Let the first plane cut AC in some point E, and AB in some point D, and likewise the second plane cut $A'C'$ and $A'B'$ in corresponding points E' and D'.

Why must there be such intersection points? Must they be between A and C, A and B, A' and C', A' and B' respectively?

Then $\measuredangle DFE$ and $D'F'E'$ are the measures of the dihedral angles whose edges are AP and $A'P'$, and these are to be proved equal to each other.

Now use the pairs of face triangles APB, $A'P'B'$, etc., to show that $\triangle ABC \cong \triangle A'B'C'$. Then show in order

(1) $\triangle ADF \cong \triangle A'D'F'$. (2) $\triangle AEF \cong \triangle A'E'F'$.

(3) $\triangle ADE \cong \triangle A'D'E'$. (4) $\triangle DFE \cong \triangle D'F'E'$.

In like manner the other pairs of dihedral angles may be proved equal to each other.

65. Two polyhedral angles are **congruent** if they can be so placed that their vertices and edges coincide.

A polyhedral angle is read by naming the vertex and one letter in each edge, as *P–ABCDE*, or by the vertex alone where no ambiguity would arise.

66. Corollary. *If two trihedral angles have their face angles equal each to each and arranged in the same order, they are congruent.*

For by the theorem their dihedral angles are equal each to each, and if the equal faces are arranged in the same order, the equal parts may be applied in succession, and the trihedral angles may be made to coincide throughout.

SUMMARY OF CHAPTER I.

1. Describe the various ways of determining a plane.

2. State the axioms used in this chapter.

3. State the definitions and theorems on perpendicular lines and planes.

4. State the definitions and theorems on parallel lines and planes.

5. Name some applications in connection with perpendiculars and parallels which have impressed you.

6. State some facts in regard to perpendiculars and parallels in the plane which do not hold in space.

7. Give the definitions and theorems on dihedral angles.

8. What theorems on perpendicular planes are proved in connection with dihedral angles?

9. Give the definitions and theorems on projections used in this chapter.

10. Give the definitions and theorems on polyhedral angles thus far used.

11. Make a list of all the loci problems in this chapter.

12. Study the following collection of problems and applications, and then make a collection of those which impress you as most interesting or useful. Include in this list any applications in the chapter.

PROBLEMS AND APPLICATIONS.

1. A Christmas tree is made to stand on a cross-shaped base. If the tree is perpendicular to each piece of the cross is it perpendicular to the floor?

2. If the projections on a plane of a number of points outside the plane lie in the same straight line, do the points themselves necessarily lie in a straight line?

3. If A, B, and C do not lie in the same line, and if their projections on a plane M do lie in a straight line, what is the relation of the planes M and ABC?

4. Is it possible to project a circle upon a plane so that the projection shall be a straight line-segment? If so, how must the circle and the plane be related?

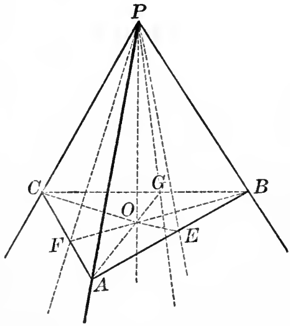
5. How many planes may be made to pass through a given point parallel to a given line? Discuss the mutual relations of all such planes.

6. Through a point P construct a line meeting each of two lines l_1 and l_2.

SUGGESTION. Let M and N be the planes determined by P and l_1 and by P and l_2. Is this construction always possible?

7. Given a plane M and lines l_1, l_2. Construct a line perpendicular to M and meeting both l_1 and l_2.

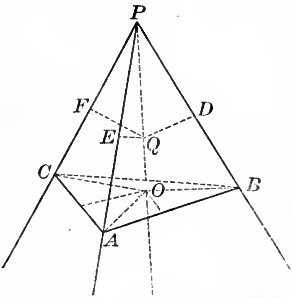
SUGGESTION. Project l_1 and l_2 on M.



8. A line l is parallel to a plane M, and lines l_1 and l_2 in M are not parallel to l. Show that the shortest distance between l and l_1 is equal to the shortest distance between l and l_2.

9. Prove that the planes bisecting the dihedral angles of a trihedral angle meet in a line.

10. Find the locus of all points equidistant from the planes determined by the faces of a trihedral angle.



11. Find the locus of all points equidistant from the edges of a trihedral angle.

SUGGESTION. On the edges lay off $PA = PB = PC$. Let O be equidistant from A, B, and C. Then any point Q in PO is equidistant from the edges.

12. Planes determined by the edges of a trihedral angle and the bisectors of the opposite face angles meet in a line.

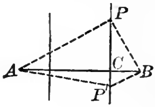
SUGGESTION. If, in the figure of Ex. 9, $PA = PB = PC$, and if PE, PF, PG bisect the face angles, then E, F, G are the middle points of the sides of the triangle ABC. Now apply the theorem that the medians of a triangle meet in a point.

13. Planes perpendicular to the faces of a trihedral angle and bisecting its face angles meet in a line.

14. Show how to locate a point which is at a distance of 2 feet from each face of a trihedral angle.

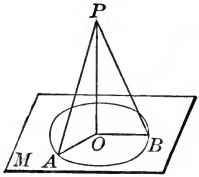
Suggestion. Pass a plane parallel to each face of the trihedral angle and at a distance of 2 feet from it.

15. Show how to locate a point which is 2 feet from one face of a trihedral angle, 3 feet from the second, and 4 feet from the third.



16. Find the locus of a point in space such that the difference of the squares of its distances from two fixed points is constant.

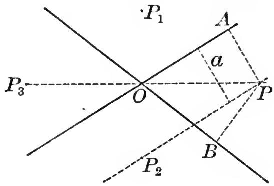
Suggestion. First solve the problem in the plane, obtaining as the required locus the line PP'. Then rotate the figure about the line AB as an axis. See Plane Geometry, page 227, Ex. 7.



17. Find the locus of all points in a plane at which lines from a fixed point P not in the plane meet the plane at equal angles.

18. Find the locus of all points which are at the same fixed distance from each of two intersecting planes.

Solution. Pass a plane perpendicular to the intersection of the two given planes, making the plane cross section shown in the figure. Draw the bisector of $\angle AOB$ and a line at the required distance a from OA and parallel to it.



Then P is at the distance a from each of the lines OA and OB. In the same manner three other such points, P_1, P_2, P_3, are constructed.

Now let this figure move through space parallel to itself, the point O moving along the intersection of the given planes. The points P, P_1, P_2, P_3 will thus move along straight lines which constitute the required locus.

19. Find the locus of points 3 feet from one of two intersecting planes and 6 feet from the other.

20. It is required that a series of electric lights shall be 7 feet above the floor of a room and 3 feet from the walls. Find the locus of all points at which such lights may be placed.

21. Find the locus of all points in space equally distant from each of two intersecting straight lines.

22. Find the locus of all points in space equally distant from two parallel lines.

23. Given two points A and B on the same side of a plane M. Determine a point P in M such that $AP + PB$ shall be a minimum.

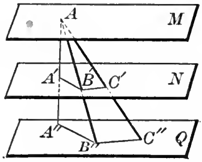
SUGGESTION. Pass a plane through A and B perpendicular to M, and proceed as in Ex. 8, page 200, Plane Geometry.

24. Show that if the edge of a dihedral angle is cut by two parallel planes, the sections which they make with the faces form equal angles.

25. Show that if all edges of a trihedral angle are cut by each of a series of parallel planes, the intersections form a series of similar triangles.

26. Find the locus of the intersection points of the medians of the triangles obtained in Ex. 25. Also of the altitudes.

27. If three planes are so related that the segments intercepted on any transversal line are in the same ratio as the segments intercepted on any other transversal then the planes are parallel.



SUGGESTION. Let M, N, Q be the three planes. From A any point in M draw three lines, not in the same plane, meeting N in A', B', C', and Q in A'', B'', C''. Use the hypothesis to show that $A'B' \parallel A''B''$ and $C'B' \parallel C''B''$. Hence $Q \parallel N$. Similarly show that $M \parallel N$.

28. A segment AB of fixed length is free to move so that its end-points lie in two fixed parallel planes. Find the locus of a point C on AB if AC is of fixed length.

29. If the projections of a set of points on each of two planes not parallel to each other lie in straight lines, show that the points themselves lie in a straight line.

30. If l_1 and l_2 are not parallel and non-intersecting, show that there is only one plane through l_1 parallel to l_2.

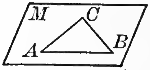
31. Show that if two lines are not parallel and do not lie in the same plane, they have only one common perpendicular, and that the shortest distance between the lines is measured along this perpendicular.

CHAPTER II.

PRISMS AND CYLINDERS.

67. Definitions. Any portion of a plane entirely bounded by line-segments or curves is called a **plane-segment**.

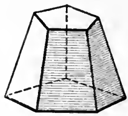
E.g. the portion of the plane M inclosed by the triangle ABC is the plane-segment ABC.



If the boundary is composed entirely of straight line-segments, the inclosed portion is called a **polygonal plane-segment**.

68. A **polyhedron** is a three-dimensional figure formed by polygonal plane-segments which entirely inclose a portion of space.

The line-segments which are common to two polygons are the **edges** of the polyhedron.



The plane-segments inclosed by the edges are the **faces**, and the intersections of the edges are the **vertices** of the polyhedron. A polyhedron is **convex** if every section of it made by a plane is a convex polygon.

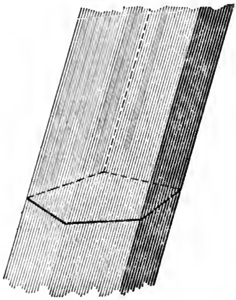
NOTE. The word *polyhedron*, as here defined, means the *surface* inclosing a portion of space and not that portion of space itself. However, it is sometimes convenient to use the word *polyhedron* when referring to the inclosed space. Thus, we speak of dividing a polyhedron into other polyhedrons when, strictly, we mean that smaller polyhedrons are constructed which divide into parts the space inclosed in the given polyhedron.

In the same manner the word *polygon* is sometimes used to indicate the plane-segment bounded by it, and the word *angle* to indicate the part of the plane within it.

Thus, a face of a polyhedron is sometimes called a polygon, and the face of a polyhedral angle is sometimes called an angle.

In all cases the context will clearly indicate what is meant.

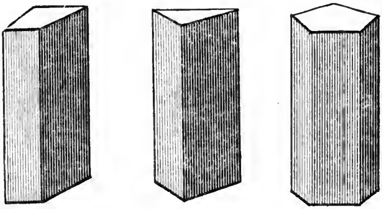
69. Definitions. Given a convex polygon and a straight line not in its plane. If the straight line move so as to remain parallel to itself while it always touches the polygon and is made to traverse it completely, the line is said to **generate a closed prismatic surface.**



The moving line is called the **generator** of the surface, and the guiding polygon the **directrix.**

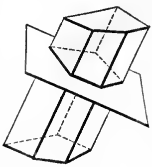
A **cross section** of a prismatic surface is made by any plane cutting it and not parallel to the generator.

70. A **prism** is that part of a closed prismatic surface included between two parallel cross sections, together with the intercepted plane-segments.



The parallel plane-segments are the **bases** of the prism, and the portion of the prismatic surface between the bases is called the **lateral surface** of the prism.

The lateral surface is composed of parallelograms (why?), and these are called the **lateral faces** of the prism. The sides of these parallelograms, not common to the bases, are the **lateral edges** of the prism.



A **right section** of a prism is made by a plane cutting each of its lateral edges, extended if necessary, and perpendicular to them.

71. Prisms are classified according to the form of their right sections, as **triangular, quadrangular, pentagonal, hexagonal,** etc. A **regular prism** is one whose right section is a regular polygon.

A prism is a **right prism** if its lateral edges are perpendicular to its bases; otherwise it is **oblique.**

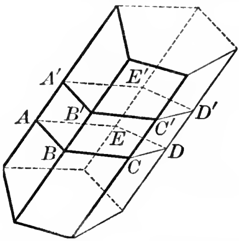
The **altitude** of a prism is the perpendicular distance between its bases. The altitude of a right prism is equal to its edge.

The **lateral area** of a prism is the sum of the areas of its lateral faces.

The **total area** is the sum of its lateral area and the area of its bases.

THEOREMS ON PRISMS.

72. THEOREM. *The cross sections of a prism made by parallel planes are congruent polygons.*



Given a prism cut by two parallel planes forming the polygons $ABCDE$ **and** $A'B'C'D'E'$.

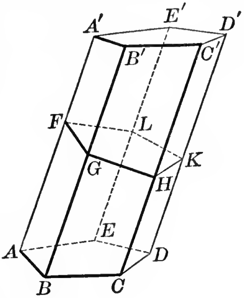
To prove that $ABCDE \cong A'B'C'D'E'$.

Outline of proof: (1) Show that $AB = A'B'$, $BC = B'C'$, etc., by proving that $ABB'A'$, $BCC'B'$, etc., are $\boxed{S}$.

(2) Show that $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$, etc.

(3) Hence, show that $ABCDE$ and $A'B'C'D'E'$ can be made to coincide.

73. THEOREM. *The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.*



Suggestion. Show that the lateral edges are mutually equal and that the area of each face is the product of a lateral edge and one side of the right-section polygon.

Complete the proof.

NOTE. The form of statement in this theorem is the usual abbreviation for the more *precise* form:

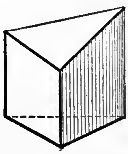
The lateral area of a prism is equal to the product of the *numerical measures* of a lateral edge and the perimeter of a right section.

Similar abbreviations are used throughout this text.

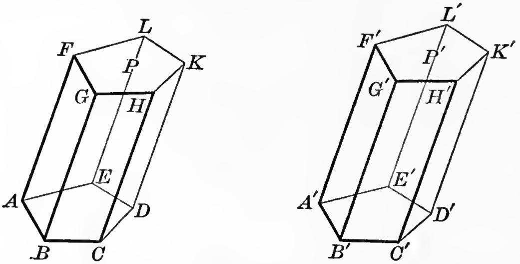
74. COROLLARY. *The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.*

75. Definitions. A polyhedron which is a part of a prism cut off by a plane meeting all the lateral edges, but not parallel to the base, is called a **truncated prism.**

Two polyhedrons are said to be **added** when they are placed so that a face of one coincides with a face of the other, but otherwise each lies outside the other.



76. THEOREM. *Two prisms are congruent if three faces having a common vertex in the one are congruent respectively to three faces having a common vertex in the other, and similarly placed.*



Given the three faces meeting at *B* in prism *P* congruent respectively to the three faces meeting at *B'* in prism *P'*, and similarly placed.

To prove that P can be made to coincide with P'.

Outline of proof: Trihedral angles B and B' are congruent. (Why?)

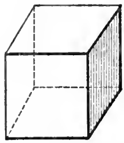
Now apply the two prisms, making B coincide with B', and thus show in detail that:

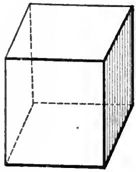
(1) The lower bases coincide.
(2) The lateral faces at B and B' coincide.
(3) The upper bases coincide.
(4) All the lateral faces coincide.

Hence P and P' coincide throughout.

77. COROLLARY. *Two right prisms are congruent if they have congruent bases and equal altitude.*

78. COROLLARY. *Two truncated prisms are congruent if the faces forming a trihedral angle of one are equal respectively to the corresponding faces of the other.*



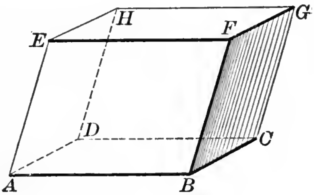


79. Definitions. A parallelopiped is a prism whose bases, as well as lateral faces, are parallelograms.

A **rectangular parallelopiped** has all its faces rectangles.

A **cube** is a parallelopiped whose faces are all squares.

80. THEOREM. *Any two opposite faces of a parallelopiped are parallel and congruent.*



Suggestions. Consider the opposite faces $ABFE$ and $DCGH$.

(1) Show that the sides of the angles ABF and DCG are parallel, and hence the planes determined by them are parallel.

(2) Show that these faces are congruent.

In like manner argue about any other pair of opposite faces.

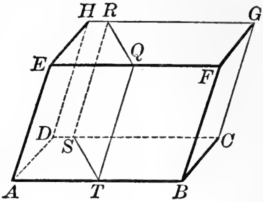
81. **EXERCISES.**

1. Can a parallelopiped be a *right* prism without being a *rectangular* parallelopiped? Illustrate.

2. Show that in a rectangular parallelopiped each edge is perpendicular to all the other edges meeting it.

3. Is it possible to construct a prism which has no right section according to the definition of § 70?

4. Show that any section of a parallelopiped made by a plane cutting four parallel edges is a parallelogram.

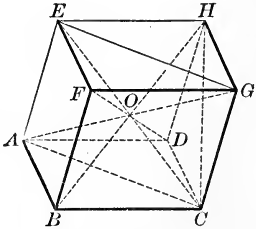


5. A section made by a plane passed through two diagonally opposite edges of a parallelopiped is a parallelogram.

E.g. the section through DH and BF in the figure of Ex. 4.

6. Show that any two of the four diagonals of a parallelopiped bisect each other.

E.g. AG and CE in the figure. Make use of the preceding example.



7. Show that the diagonals of a rectangular parallelopiped are all equal to each other.

8. Show that the square on the diagonal of a rectangular parallelopiped is equal to the sum of the squares on the sides meeting at a vertex from which it is drawn.

E.g. in the figure of Ex. 6, $\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$.

9. Find the ratio of the diagonal to one edge of a cube.

10. Find the edge of a cube whose diagonal is 14 inches. Find the diagonal of a cube whose edge is 16 inches.

11. Find a diagonal of a rectangular parallelopiped whose edges are 6, 8, and 10 inches respectively.

12. Are the diagonals of a cube perpendicular to each other?

13. If two diagonals of a rectangular parallelopiped meet at right angles, and if two of its faces are squares, find the ratio of the sides of the remaining faces.

14. If two congruent right prisms whose bases are equilateral triangles are placed together so as to form one prism whose base is a parallelogram, compare the lateral area of the prism so formed with the sum of the lateral areas of the original prisms.

15. A right prism whose bases are regular hexagons is divided into six prisms whose bases are equilateral triangles. Compare the lateral area of the original prism with the sum of the lateral areas of the resulting prisms.

16. If the perimeter of a right section of a prism is 24 inches and its altitude 6 inches, what is the smallest possible lateral area? What can be said about the largest possible area of such a prism?

17. Any section of a prism made by a plane parallel to a lateral edge is a parallelogram.

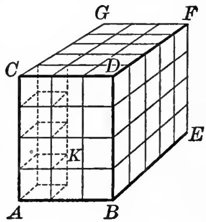
18. Show that for every prism there is at least one set of parallel planes which cut the prism in rectangular sections. See suggestions Ex. 7, p. 49.

VOLUMES OF RECTANGULAR PARALLELOPIPEDS.

82. Thus far certain properties of prisms have been studied, but no attempt has been made to *measure the space inclosed* by such a solid. For this purpose we consider first a rectangular parallelopiped.

83. Definition. In case each edge of a rectangular parallelopiped is commensurable with a unit segment, the number of times which a unit cube is contained in it is the **numerical measure** or the **volume** of the parallelopiped.

84. In the case just described in the definition, the volume is easily computed.



E.g. if in the figure one edge AC is 4 units, and an adjoining edge AB is 3 units, then a cube as AK, whose edge is one unit, may be laid off 4 times along AC and a tier of $3 \cdot 4 = 12$ such cubes will adjoin the face AD, while 5 such tiers will exactly fill the space within the solid. That is, $5 \cdot 3 \cdot 4 = 60$ is the number of cubic units in the solid.

Again, if the given dimensions are 3.4, 2.6, 4.5 decimeters respectively, then unit cubes, with edge one *decimeter*, cannot be made to fill exactly the space inclosed by the figure, but cubes with edge each one *centimeter* will do so, giving 34, 26, and 45 respectively along the three edges of the solid.

Hence the volume is

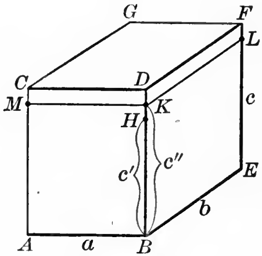
$$34 \cdot 26 \cdot 45 = 39{,}780 \text{ cubic centimeters, or } 39.78 \text{ cubic decimeters.}$$

85. Axiom IV. *Any rectangular parallelopiped incloses a definite volume which is greater than that of another, provided no dimension of the first is less than the corresponding dimension of the second, and at least one dimension is greater.*

86. THEOREM. *The volume of any rectangular parallelopiped is equal to the product of the numerical measures of its linear dimensions.*

Proof: CASE 1. *If each dimension is commensurable with the unit segment.*

This is the case treated in § 84.



CASE 2. *If two dimensions are commensurable with the unit segment and the third is not.*

By Axiom IV the parallelopiped has a definite volume V.

Suppose this is not equal to abc and that $V < abc$.

Let c' be a number such that $V = abc'$. Then $c' < c$. On BD lay off $BH = c'$.

Divide the unit segment into equal parts, each less than HD, and lay off one of these parts successively on BD, reaching a point K between H and D. Denote the length of BK by c'' and pass a plane through K parallel to ABE.

By Case 1 the volume of the parallelopiped AL is abc''.
By hypothesis, $V = abc'$, but $abc' < abc''$, since $c' < c''$.
Hence, $V < abc''$. (1)
But by Ax. IV, $V > abc''$. (2)
Hence, the assumption that $V < abc$ cannot hold.
In the same manner show that $V > abc$ cannot hold.
Hence, we have $V = abc$.

CASE 3. *If one dimension is commensurable with the unit segment and two are not.*

CASE 4. *If all three dimensions are incommensurable with the unit segment.*

The proofs in these cases are similar to the above.

Thus, in Case 3, when b and c are both incommensurable with the unit segment, we obtain the parallelopiped AL, two of whose dimensions a and c'' are commensurable with the unit, and hence, by Case 2, its volume is abc''. Then the argument is identical with that given before. **Hence, in all cases $V = abc$.**

87. COROLLARY 1. *The volume of a rectangular parallelopiped is equal to the product of the numerical measures of its base and altitude.*

88. COROLLARY 2. *If two rectangular parallelopipeds have two dimensions respectively equal to each other, their volumes are in the same ratio as their third dimensions; and if they have one dimension the same in each, their volumes are in the same ratio as the products of their other two dimensions.*

For if V and V' are the volumes, and a, b, c and a', b', c' the dimensions, then $\frac{V}{V'} = \frac{a \cdot b \cdot c}{a' \cdot b' \cdot c'} = \frac{a}{a'}$ if $b = b'$ and $c = c'$; or $\frac{V}{V'} = \frac{a \cdot b}{a' \cdot b'}$ if $c = c$.

VOLUMES OF PRISMS IN GENERAL.

89. From the formula for rectangular parallelopipeds,

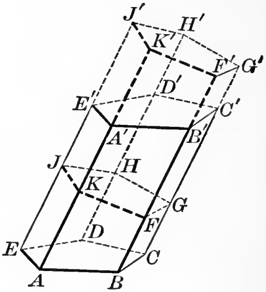
Volume = length × width × altitude, or $V = abc$

we deduce the volumes of prisms in general by means of the principle:

Two polyhedrons are equal (that is, have the same volume) if they are congruent, or if they can be divided into parts which are congruent in pairs.

The sign = between two polyhedrons means that they are equal in volume. The word *equivalent* is sometimes used to mean *equal in volume.*

90. THEOREM. *The volume of an oblique prism is equal to that of a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism.*



Given the oblique prism AD', with $FGHJK$ a right section, and $F'G'H'J'K'$ a right section of the prism extended so that the edge $AA' = KK'$.

To prove that the oblique prism AD' has the same volume as the right prism KH'.

Outline of proof: Show that

(1) $ABCDE \cong A'B'C'D'E'$;

(2) $ABFK \cong A'B'F'K'$;

(3) $EAKJ \cong E'A'K'J'$.

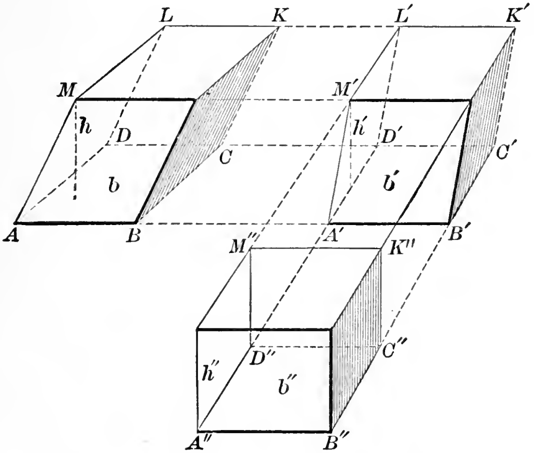
Hence, by § 78 AH and $A'H'$ are congruent.

Now the given prism $AD' = AH + KD'$ (§ 75)

and the right prism $KH' = A'H' + KD'$.

Hence, $AD' = KH'$. (Why?)

91. THEOREM. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Given the oblique parallelopiped AK, with base b and alt. h.

To prove that its volume is equal to b times h.

Proof: Considering face AL as the base of prism AK, produce the four edges parallel to AB, and lay off $A'B' = AB$.

Through A' and B' erect planes $\perp$ to AB', thus cutting off the right prism $A'K'$ with $A'L'$ as one base.

Now considering $C'L'$ as the base of prism $A'K'$, produce the four edges $\parallel$ to $C'B'$ and lay off $C''B'' = C'B'$.

At C'' and B'' erect planes $\perp$ to $C'B''$, cutting off the right prism $A''K''$, which is a rectangular parallelopiped.

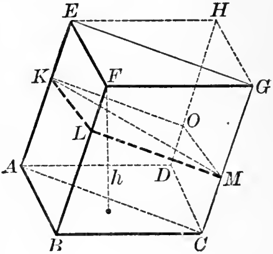
Then show that (1) $h = h' = h''$; (2) $b = b' = b''$; (3) prisms AK, $A'K'$, $A''K''$ are equal (§ 90).

But prism $A''K'' = b'' \cdot h''$. (Why?) Hence, prism $AK = b \cdot h$.

Write out this proof in full.

92. THEOREM. *The volume of a triangular prism is equal to the product of its base and altitude.*

Given the triangular prism whose base is ABC.



To prove that the volume of this prism is equal to the area of $\triangle\ ABC$ multiplied by the altitude h, that is, by the perpendicular distance between ABC and EFG.

Proof: Complete the ▱s $ABCD$ and $EFGH$ and draw DH. Now show that $CDHG$ and $ADHE$ are ▱s, and hence that BH is a parallelopiped.

Let $KLMO$ be a right section of BH, and let KM be the line in which the plane $ACGE$ cuts the plane $KLMO$.

Now (1) $KLMO$ is a ▱. Hence, $\triangle\ KLM \cong \triangle\ KMO$.

(2) Volume of prism $ABC\text{–}F =$ area $\triangle\ KLM$ times BF.

(3) Volume of prism $BH =$ area ▱ $KLMO$ times BF.

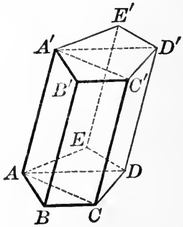
(4) Hence, prism $ABC\text{–}F = \frac{1}{2}$ prism BH.

(5) But prism $BH = h \times$ area of ▱ $ABCD$.

(6) Hence, prism $ABC\text{–}F = h \times \frac{1}{2}$ area of ▱ $ABCD$ $= h \times$ area of $\triangle\ ABC$.

Therefore prism $ABC\text{–}F$ is equal to the product of its base and altitude. Give reasons for each step.

93. COROLLARY 1. *The volume of any prism is equal to the product of its base and altitude.*



For any prism can be divided into triangular prisms by planes passing through one edge and each of the other non-adjacent edges.

94. Corollary 2. *Any two prisms of equal altitudes have the same volumes if their bases are equal.*

95. Corollary 3. *The volumes of two prisms have the same ratio as their altitudes if their bases are equal; and the same ratio as the areas of their bases if their altitudes are equal.*

96. **EXERCISES.**

1. The proposition that the volume of any prism is equal to the product of its base and altitude is of great importance. What theorems of this chapter were used directly or indirectly in proving it?

2. If the base of a prism is 36 square inches and its altitude 12 inches, what is its volume?

3. If the perimeter of the base of a prism and the length of a lateral edge are known, is the lateral area thereby determined?

4. If the area of the base and the length of an edge are known, can the volume be found?

5. What dimensions of a prism must be known in order to determine its lateral area by means of the theorems of this chapter? What dimensions must be known to determine its volume?

6. Parallel sections of a closed prismatic surface are congruent polygons. Prove.

7. Find the edge of a cube if its total area is equal numerically to its volume, an inch being used as a unit.

8. A side of the base of a regular right hexagonal prism is 3 inches. Find its altitude if its volume is $54\sqrt{3}$ cubic inches. What is the total area of this prism?

9. A side of one base of a regular right triangular prism is equal to the altitude of the prism. Find the length of the side if the total surface is numerically equal to the volume.

10. Solve the preceding problem in case the prism is a regular right hexagonal prism.

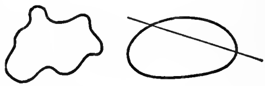
11. The volume of a regular right prism is equal to the lateral area multiplied by half the apothem of the base. Prove.

CYLINDERS.

97. Definitions. A surface no segment of which is plane is called a **curved surface.**

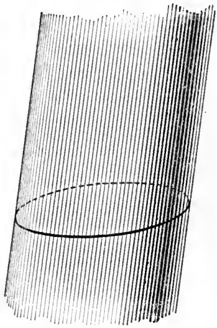
E.g. the surface of an eggshell or of a stovepipe is a curved surface.

A **closed plane curve** is one which can be traced continuously by a point moving in a plane so as to return to its original position without crossing its path.



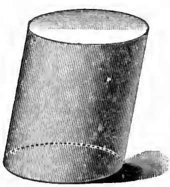
A **convex closed curve** is one which can be cut by a straight line in only two points.

98. Given a closed convex plane curve and a straight line not in its plane. If the straight line moves so as to remain parallel to itself, while it always touches the curve and is made to traverse it completely, the line is said to generate a **closed convex cylindrical surface.** The moving line is the **generator**, and the guiding curve the **directrix.** The generator in any one of its positions is an **element** of the surface.



A **cross section** of a cylindrical surface is made by a plane cutting all its elements.

99. A **cylinder** is that part of a closed cylindrical surface included between two parallel cross sections, together with the plane-segments thus intercepted. The plane-segments are the **bases** of the cylinder, and the part of the cylindrical surface between the bases is the **lateral surface.**



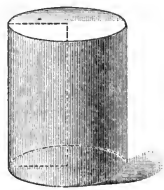
The portion of the generating line in any position which is included between the bases is an **element** of the cylinder.

A **right section of a cylinder** is made by a plane cutting each of its elements at right angles.

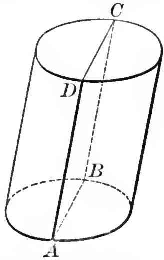
A **circular cylinder** is one whose right section is a circle.

The radius of a circular cylinder is the radius of its right section.

A cylinder whose bases and right sections are circles is called a **right circular cylinder.** A right circular cylinder is called a **cylinder of revolutions**, since it may be generated by revolving a rectangle about one of its sides as an axis. The side opposite the axis generates the lateral surface, and the sides adjacent to the axis generate the bases.



100. THEOREM. *If a plane contains an element of a cylinder and meets it in one other point, then it contains another element also, and the section is a parallelogram.*



Given the cylinder *AC* and a plane containing the element *AD* and one other point as *B*.

To prove that it contains another element *BC* and that *ABCD* is a parallelogram.

Proof: Through *B* pass a line *BC* ∥ *AD*.

Then *BC* is an element of the cylinder. (Why?)

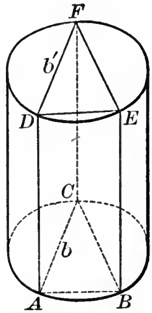
Also *BC* lies in the plane *ABD*.

Now show that *ABCD* is a parallelogram.

What is the section *ABCD* if *AC* is a *right* cylinder?

101. Definition. If a plane contains an element of a cylinder and no other point of it, the plane is said to be **tangent** to the cylinder, and the element is called the **element of contact.**

102. THEOREM. *The bases of a cylinder are congruent plane-segments.*



Given a cylinder with the bases *b* and *b′*.

To prove that $b \cong b'$.

Proof: Take any three points A, B, C in the rim of the base b and draw the elements at these points, meeting the base b' in D, E, F.

Show that $\triangle ABC \cong \triangle DEF$.

Now, while the elements AD and BE remain fixed, conceive CF to generate the cylinder.

Evidently $\triangle ABC \cong \triangle DEF$ in every position of CF.

Hence, if base b' is applied to base b with these triangles coinciding in *one* position, they will coincide in every position corresponding to the moving generator.

That is, $b \cong b'$.

103. EXERCISES.

1. Show that right sections of any cylinder are congruent, and any section parallel to the base is congruent to the base.

2. If two intersecting planes are each tangent to a cylinder, show that their line of intersection is parallel to an element of the cylinder and also parallel to the plane containing the two elements of contact.

3. What is the locus of all points at a perpendicular distance of 2 feet from a given line?

4. If the section of a cylinder made by every plane parallel to an element of it is a rectangle, what kind of cylinder is it?

5. If the radius r of a right circular cylinder is equal to its altitude, find the distance from the center of the base to a plane whose intersection with the cylinder is a square.

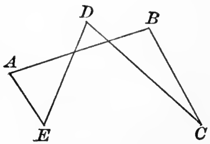
6. Roll a sheet of paper so as to form a circular cylinder, that is, one whose *right section is a circle.* Now determine by inspection the shape of the base if the paper is cut so as to let the cylinder stand in an oblique position. Also deform the cylinder so as to make the oblique base circular, and then determine the shape of the right section.

7. Show that for every cylinder there is at least one set of parallel planes which cut the cylinder in rectangular sections.

Suggestion. Project an element on the plane of the base, and draw lines in the base at right angles to this projection. Through these lines pass planes parallel to an element.

8. Is a polygon circumscribed about a convex closed curve necessarily a convex polygon? See § 160, Plane Geometry.

9. Is a polygon inscribed in a convex closed curve necessarily convex? (Note that a broken line AB, BC, CD, DE, EA which cuts itself as shown in the figure is not here regarded as a polygon.) The answers to Exs. 8 and 9 limit the polygons to be used under Ax. V to convex polygons. Hence it is not necessary to state in that axiom that these polygons must be convex.



10. If polygons other than convex are permitted, is it possible to construct one within a convex closed curve whose perimeter is greater than the length of the curve? Can such polygons be inscribed in the curve?

MEASUREMENT OF THE SURFACE AND VOLUME OF A CYLINDER.

104. Thus far in geometry the word **area** has been used in connection with *plane-segments only.* In some cases the computation of an area has been possible by an approximation process only, as in the case of some rectangles and of the circle.

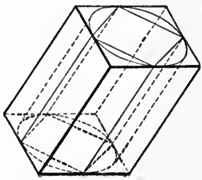
In the case of *any curved surface* it is evident that approximate measurement is the *only kind possible* in terms of a plane area unit, since no such unit, however small, can be made to coincide with a segment of a curved surface.

Theorems concerning the surface and the volume of a cylinder are based upon the following definitions and assumptions:

105. Axiom V. *Any convex closed curve has a definite length and incloses a definite area, which are less respectively than the perimeter and area of any circumscribed polygon and greater than those of any inscribed polygon.*

Also the perimeter and area of either the inscribed or the circumscribed polygon may be made to differ from those of the curve by as little as we please by taking all the sides sufficiently small.

106. Definitions. A prism is said to be **inscribed in a cylinder** if its lateral edges are elements of the cylinder, and their bases lie in the same planes.



A prism is said to be **circumscribed about a cylinder** if its lateral faces are all tangent to the cylinder, and their bases lie in the same planes.

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107. Axiom VI. *The lateral surface of a convex cylinder has a definite area, and the cylinder incloses a definite volume, which are less respectively than those of any circumscribed prism and greater than those of any inscribed prism.*

108. THEOREM. *The lateral area of a convex cylinder is equal to the product of an element and the perimeter of a right section.*

Given a convex cylinder of which L is the lateral area, e an element, and p the perimeter of a right section.

To prove that $L = ep$.

Proof: It will be shown that L can be neither greater than nor less than ep.

First, suppose $L < ep$.

Let $L = eK$. Then $K < p$ (1)

Now inscribe a cylinder the perimeter p' of whose right section is greater than K. This is possible by § 105.

Then $ep' > eK$. (2)

Hence, ep' is greater than the supposed value eK of L.

But this is impossible, since ep' is the area of an *inscribed* prism (§ 107).

Hence, the supposition $L < ep$ leads to a contradiction.

Secondly, the supposition that $L > ep$ may be shown to be impossible by using a circumscribed prism.

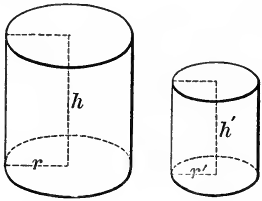
Hence, we have $L = ep$.

109. COROLLARY. *If r is the radius of any circular cylinder, and e is an element, then $L = 2\,\pi re$.*

In the case of a *right* circular cylinder $e = h$, the altitude, and we have $L = 2\,\pi rh$.

110. Definition. Two right circular cylinders are **similar** if they are generated by similar rectangles revolving about corresponding sides.

111. THEOREM. *The lateral areas or the entire areas of two similar right circular cylinders are in the same ratio as the squares of their altitudes or as the squares of the radii of their bases.*



Suggestion. If h and h' are the altitudes, r and r' the radii, L and L' the lateral areas, and A and A' the total areas, we are to show that

$$\frac{L}{L'} = \frac{A}{A'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}.$$

Make use of the following, giving each in detail.

$$(1)\ L = 2\,\pi r h, \quad (2)\ A = 2\,\pi r(r+h),$$

$$(3)\ \frac{r}{r'} = \frac{h}{h'}, \text{ and } (4)\ \frac{r+h}{r'+h'} = \frac{r}{r'} = \frac{h}{h'}.$$

112. THEOREM. *The volume of any convex cylinder is equal to the product of its altitude and the area of its base, or to the product of an element and the area of its right section.*

Suggestions. (1) If h is the altitude and b the base, show by an argument similar to that of § 108, that V, the volume, can be neither greater nor less than bh.

(2) If e is an element and c the area of a right section, give a similar argument to show that $V = ec$, using § 90. Give all the steps in full.

113. Corollary. *If a cylinder has a right circular section whose radius is* r_1 *and an element* e, *then* $V = \pi r_1^2 e$.

If a cylinder has a circular base of radius r_2, *and an altitude* h, *then* $V = \pi r_2^2 h$.

In the case of a right circular cylinder, $r_1 = r_2$ and $h = e$. Hence, the two formulas become identical.

114. Theorem. *The volumes of two similar right circular cylinders are in the same ratio as the cubes of the radii of their bases or as the cubes of their altitudes.*

Give the proof in full, using suggestions similar to those given in § 111.

PROBLEMS AND APPLICATIONS.

1. What dimensions of a cylinder must be known in order that its lateral area may be computed? State fully for different kinds of cylinders.

2. What dimensions of a cylinder must be known in order that its volume may be computed?

3. If the base of a cylinder is a circle with radius 5 inches, find its volume if the altitude is 8 inches.

4. If the lateral surface of a cylinder and the length of an element are known, can the perimeter of a right section be found? If the lateral area is 400 π, and an element 15, find the perimeter of a right section.

5. The diameter of a right circular cylinder is 8, and the diagonal of the largest rectangle which can be cut from it is 16. Find its altitude.

6. The volume of a right circular cylinder is 128 π cubic inches and its altitude is equal to its diameter. Find the altitude and the diameter.

7. If the diameter of a right circular cylinder is equal to its altitude, determine the diameter so that the total area of the cylinder is equal numerically to its volume.

8. Find the diameter of a right circular cylinder if the total area of the inscribed regular triangular prism is equal numerically to the volume of the cylinder, the diameter of the cylinder being equal to its altitude.

9. In the preceding find the diameter of the cylinder if the volume of the prism is equal numerically to the total area of the cylinder.

10. Solve a problem like Ex. 8 if a regular hexagonal prism is used instead of a triangular prism.

11. Solve a problem like Ex. 9 if a regular hexagonal prism is used instead of a triangular prism.

12. A rectangle whose sides are a and b is turned about the side a as an axis and then about the side b. Find the ratio of the volumes of the two cylinders thus developed.

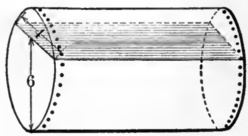
13. Compare the total surfaces of the two figures developed in Ex. 12.

14. Find the diameter of a right circular cylinder if its lateral area is equal numerically to its volume. Does the result depend upon the altitude of the cylinder?

15. If the altitude of a right circular cylinder is equal to its diameter, find the ratio of the numerical values of its total area and its volume. Does this depend on the radius?

16. A regular octagonal prism is inscribed in a right circular cylinder whose altitude is equal to the diameter. Find the difference between the volumes of the cylinder and the prism, if a side of the octagon is 4 inches.

17. A cylindrical tank 8 feet in diameter, partly filled with water, is lying on its side. If the greatest depth of the water is 6 feet, what fraction of the volume of the tank is filled with water?

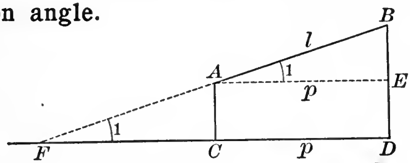


18. In the preceding problem find the fraction of the volume occupied by water if the width of the top of the water along a right cross section of the tank is 4 feet.

THEOREMS ON PROJECTION.

115. The projection of a line-segment on a given line was defined in § 57. The **length of the projection** will now be computed in terms of the given line-segment.

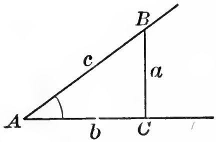
116. Definitions. The acute angle between a line-segment and a given line on which it is projected is called the **projection angle.**



E.g. $\angle AFC$ or its equal $\angle BAE$ found by drawing $AE \parallel CD$.

If l is the length of a line-segment and p the length of its projection, then the ratio $\frac{p}{l}$ is called the **cosine of the projection angle.**

E.g. in the figure, $\frac{p}{l} =$ cosine $\angle BAE$.



117. In any right triangle ABC, either acute angle, as $\angle A$, is the projection angle between the hypotenuse and the side adjacent to the angle.

Hence the **cosine of an acute angle of a right triangle** is the *ratio of the adjacent side to the hypotenuse.*

We have already defined the **sine of an acute angle of a right triangle** as the *ratio of the opposite side to the hypotenuse.* See § 279, Plane Geometry.

We now define the **tangent of an acute angle of a right triangle** as the *ratio of the opposite side to the adjacent side.*

Using the common abbreviations, sin, cos, and tan, we have in the figure:

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}.$$

118. The sine, cosine, and tangent are of great importance in many computations. By careful measurement (and in other ways) their values may be computed for any acute angle, and a table formed, like that on page 57.

E.g. if in the figure of § 117 $\angle A = 35°$ (measured with a protractor), we may measure AC, AB, and BC, and thus compute the ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$, and find the values of sin 35°, cos 35°, tan 35°.

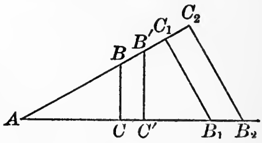
With an ordinary ruler it will not usually be possible to make these measurements with sufficient accuracy to obtain more than one decimal place.

Draw an angle of 35°, and make the measurements and computations, using an hypotenuse of various lengths (the longer the better), and show that the results do not depend upon the length of hypotenuse chosen.

119. EXERCISES.

1. Using a protractor, construct angles of 10°, 30°, 50°, 70°, and by measurement determine the sine, cosine, and tangent of each.

2. Prove that the cosine of any given angle is the same, no matter what point is taken in *either* side from which to let fall the perpendicular to the other side. Prove the same for the tangent.



3. Show that if the hypotenuse be taken *one decimeter* in length, then the length of the side adjacent, measured in decimeters, is the *cosine* of the angle, and the length of the side opposite is the *sine* of the angle.

4. Show that if the side adjacent be taken one decimeter in length, the length of the side opposite, measured in decimeters, is the *tangent* of the angle.

5. Without any direct measurement, show how to compute the three ratios for each of the angles, 30°, 45°, 60°.

SUGGESTION. Make use of the fact that if one acute angle in a right triangle is 30°, the side opposite it is one half the hypotenuse.

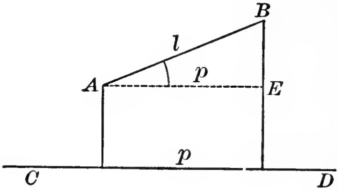
6. As the angle is made smaller and smaller, what are the values approached by the sine, cosine, and tangent?

7. As the angle is made more and more nearly 90°, what are the values approached by the sine and cosine? Discuss this case for the tangent.

TABLE OF SINES, COSINES, AND TANGENTS.

Ang.	Sin	Cos	Tan	Ang.	Sin	Cos	Tan	Ang.	Sin	Cos	Tan
0°	0	1	0	31°	.515	.857	.601	61°	.875	.485	1.80
1°	.017	1.000	.017	32°	.530	.848	.625	62°	.883	.469	1.88
2°	.034	.999	.035	33°	.545	.839	.649	63°	.891	.454	1.96
3°	.052	.999	.052	34°	.559	.829	.675	64°	.899	.438	2.05
4°	.070	.998	.070	35°	.574	.819	.700	65°	.906	.423	2.14
5°	.087	.996	.087	36°	.588	.809	.727	66°	.914	.407	2.25
6°	.105	.995	.105	37°	.602	.799	.754	67°	.921	.391	2.36
7°	.122	.993	.123	38°	.616	.788	.781	68°	.927	.375	2.48
8°	.139	.990	.141	39°	.629	.777	.810	69°	.934	.358	2.61
9°	.156	.988	.158	40°	.643	.766	.839	70°	.940	.342	2:75
10°	.174	.985	.176	41°	.656	.755	.869	71°	.946	.326	2.90
11°	.191	.982	.194	42°	.669	.743	.900	72°	.951	.309	3.08
12°	.208	.978	.213	43°	.682	.731	.933	73°	.956	.292	3.27
13°	.225	.974	.231	44°	.695	.719	.966	74°	.961	.276	3.49
14°	.242	.970	.249	45°	.707	.707	1.00	75°	.966	.259	3.73
15°	.259	.966	.268	46°	.719	.695	1.04	76°	.970	.242	4.01
16°	.276	.961	.287	47°	.731	.682	1.07	77°	.974	.225	4.33
17°	.292	.956	.306	48°	.743	.669	1.11	78°	.978	.208	4.70
18°	.309	.951	.325	49°	.755	.656	1.15	79°	.982	.191	5.14
19°	.326	.946	.344	50°	.766	.643	1.19	80°	.985	.174	5.67
20°	.342	.940	.364	51°	.777	.629	1.23	81°	.988	.156	6.31
21°	.358	.934	.384	52°	.788	.616	1.28	82°	.990	.139	7.12
22°	.375	.927	.404	53°	.799	.602	1.33	83°	.993	.122	8.14
23°	.391	.921	.424	54°	.809	.588	1.38	84°	.995	.105	9.51
24°	.407	.914	.445	55°	.819	.574	1.43	85°	.996	.087	11.43
25°	.423	.906	.466	56°	.829	.559	1.48	86°	.998	.070	14.30
26°	.438	.899	.488	57°	.839	.545	1.54	87°	.999	.052	19.08
27°	.454	.891	.510	58°	.848	.530	1.60	88°	.999	.035	28.64
28°	.469	.883	.532	59°	.857	.515	1.66	89°	1.000	.017	57.29
29°	.485	.875	.554	60°	.866	.500	1.73	90°	1.000	0	
30°	.500	.866	.577								

120. THEOREM. *The length of the projection of a line-segment upon a given line is equal to the length of the line-segment multiplied by the cosine of the projection angle.*



Given the projection p of the line-segment l on the line CD, with the projection angle A.

To prove that $p = l \cos A$.

Proof: By definition we have $\frac{p}{l} = \cos A$.

Hence, $p = l \cos A$.

121. EXERCISES.

1. Find the cosines of the angles 35° 30′, 54° 15′, 15° 45′.

SUGGESTION. The cosine of 35° 30′ lies between cos 35° and cos 36°. We assume that it lies halfway between these numbers. This assumption, while not quite correct, is very nearly so for small differences of angles, as in this case, where the total difference is only one degree. From the table cos 35° = .819, cos 36° = .809.

The number midway between these is .814, which we take as the cosine of 35° 30′.

This process is called **interpolation.** A similar process is used for sines and tangents.

2. Find the tangents of the angles 25° 20′, 47° 45′, 63° 40′.

3. Find the angle whose tangent is 1.74.

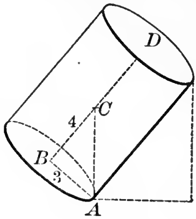
SOLUTION. From the table we have tan 60° = 1.73 and tan 61° = 1.80. Hence, the required angle must lie between 60° and 61°. Moreover, the number 1.74 is one seventh the way from 1.73 to 1.80. Hence, we assume the angle to lie one seventh the way from 60° to 61°, which gives $60° + \frac{1}{7} \times 1° = 60° + 9'$ nearly. The required angle is 60° 9′.

4. Find the angles whose sines are .276; .674; .437.

5. Find the angles whose cosines are .940; .094; .435.

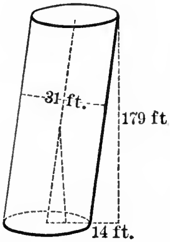
6. Find the angles whose tangents are .781; 1.41; 3.64.

Notice that as an angle increases, its sine and tangent both *increase*, but its cosine *decreases*.



7. At what angle with the horizontal must the base of a right circular cylinder be tilted to make it just topple over if its diameter is 6 feet and its altitude 8 feet?

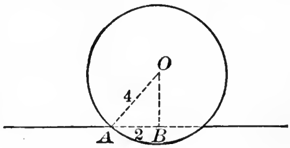
SUGGESTION. The center of gravity is at the middle point C of the axis of the cylinder. The base must be tilted so that the line AC becomes vertical.



8. The Leaning Tower of Pisa is 179 feet high and 31 feet in diameter. It now leans so that a plumb line from the top on the lower side reaches the ground 14 feet from the base.

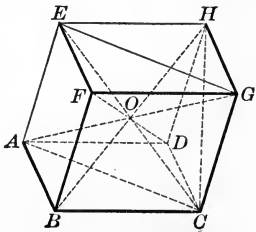
At what angle is its side now inclined from the vertical? At what angle would its side have to incline from the vertical before it would topple over?

9. A four-inch hole is cut in a board, and a ball 8 in. in diameter is made to rest on it. At what angle must the board be held so that the ball will just roll out of the hole?



SUGGESTION. The board must be held so that the line OA becomes vertical; that is, the board must be tipped at an angle equal to $\angle BOA$.

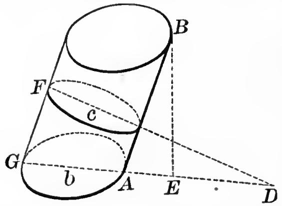
10. Using a ball 8 inches in diameter, what must be the radius of the hole in the board of the preceding problem so that the ball shall just roll out when the board is inclined at an angle of 45° to the horizontal?



11. If the figure $ABCD\text{–}H$ is a cube, find each of the following angles: $\angle ECA$, $\angle AEC$.

Check the results found by using the fact that the sum of the angles of a triangle is 180°.

122. THEOREM. *The altitude of an oblique prism or cylinder is equal to an element multiplied by the cosine of the angle between the plane of the base and that of a right section.*



Given the oblique cylinder with base *b* and right section *c*, and let *BE* be a perpendicular between the bases.

Consider the plane determined by BE and the element AB.

This plane is $\perp$ to the plane of b and also to the plane of c. (Why?)

Hence, it is $\perp$ to the line of intersection of the planes of b and c. (Why?)

Let this plane cut the planes of b and c in GD and FD respectively, D being the point in their line of intersection.

Then $\angle D$ is the measure of the dihedral angle between the planes of b and c. (Why?)

To prove that $BE = AB \cdot \cos D$.

Proof: We have $BE = AB \cdot \cos \angle ABE$. (Why?)

But $\angle D = \angle ABE$. (Why?)

Hence, $BE = AB \cdot \cos D$.

The argument is similar for any oblique prism.

123. COROLLARY. *The dihedral angle between the planes of the base and a right section of an oblique cylinder or prism is equal to the angle between an element and the altitude.*

124. **EXERCISES.**

1. Given a line-segment 10 inches long. Find the length of its projection on a plane if the projection angle is 20°. If the angle is 30°, 45°, 60°, 90°, 0°.

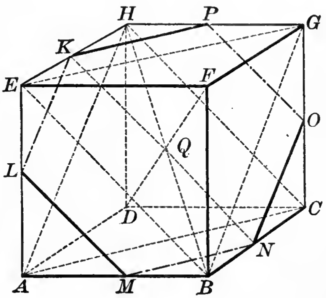
2. A kite string forms an angle of 40° with the ground. The distance from the end of the string to a point directly beneath the kite is 200 ft. Find the length of the string and the perpendicular height of the kite.

3. The altitude of an oblique prism is 15 inches. Find the length of an element if it makes an angle of 45° with the perpendicular between the bases.

4. A right section of a cylinder makes an angle of 20° with the plane of the base. Find the ratio between the altitude and an element.

5. Show that the theorem of § 122 holds for the special case of a right prism or cylinder.

6. Prove that by joining the middle points of six edges of a cube, as shown in the figure, a regular hexagon is formed.



7. Prove that in the preceding example the plane of the regular hexagon, $KLMNOP$, is perpendicular to the diagonal DF.

8. How large a cube will be required from which to cut a stopper for a hexagonal spout, each of whose sides is 4 inches?

9. In the figure find the angle KQH.

SUGGESTION. Let a be a side of the cube. Compute KH, KQ, and HQ in terms a. Note that $\angle QKH =$ rt. $\angle$.

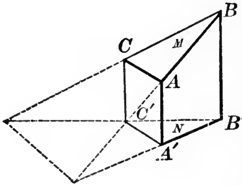
10. Find the area of the projection of the hexagon $KLMNOP$ on the face $BCGF$. Note that this projection equals the whole square less $\triangle NCO + \triangle KEL$. See § 125.

11. Find the area of the hexagon in terms of the side a of the cube.

12. By means of the theorem of § 126 and the results in Exs. 10 and 11 find the dihedral angle formed by the planes BCG and MOK.

PROJECTION OF A PLANE-SEGMENT.

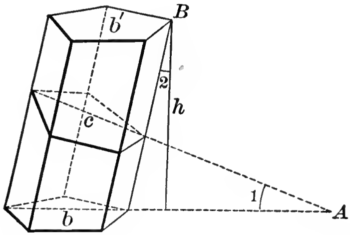
125. **Definition.** If from each point in the boundary of a plane-segment a perpendicular is drawn to a given plane, the locus of the feet of these perpendiculars will bound a portion of the plane, which is called the **projection of the plane-segment** on the given plane.



E.g. the plane-segment $A'B'C'$ in the plane N is the projection of the plane-segment ABC from the plane M upon N.

The **angle of projection** is the angle between the planes M and N.

126. THEOREM. *The area of the projection of a plane-segment on a plane is equal to the area of the plane-segment multiplied by the cosine of the projection angle.*



Proof: Let the boundary of the given plane-segment b be any convex polygon or closed curve.

Using this polygon or curve as a *directrix* and a line perpendicular to the given plane as a *generator*, develop a prismatic or cylindrical surface. The given plane will cut this surface in a right section whose area we denote by c.

Now cut the surface by a plane parallel to b, forming the upper base b' of a prism or cylinder whose altitude is h, edge e, and volume V.

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Then c is the projection of b upon the given plane, and $\angle 1 = \angle 2$ is the projection angle.

We are to show that $c = b \cos \angle 1$.

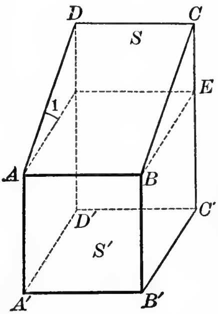
We know that $\quad V = ce = bh.$ (Why?)

But $\quad h = e \cos \angle 2.$

Hence, $\quad ce = be \cos \angle 2.$

That is, $\quad c = b \cos \angle 2 = b \cos \angle 1.$

NOTE. The foregoing theorem may be proved *directly* in case the plane-segment is a rectangle with one side parallel to the line of intersection of the two planes. In the figure let S be the given rectangle and S' its projection, with $AB \parallel$ to the line of intersection of the planes in which S and S' lie, and $\angle 1$ the angle between them.



Then $\quad S = AB \cdot BC$

and $\quad S' = A'B' \cdot B'C'.$

But $\quad AB = A'B'$

and $\quad BE = B'C'.$ (Why?)

But $\quad BE = BC \cdot \cos \angle 1$ (Why?)

and $\quad S' = A'B' \cdot B'C' = AB \cdot BC \cos \angle 1.$

That is, $\quad S' = S \cos \angle 1.$

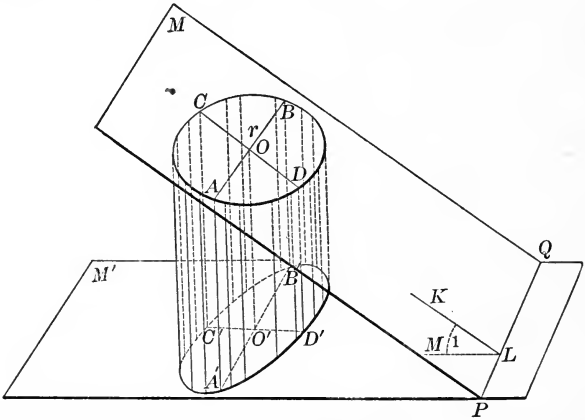
In the case of any plane-segment, rectangles may be inscribed in it in this position and their number increased indefinitely, so that their sum will approach more and more nearly to the area of the plane-segment, and in this way it may be shown to any desired degree of approximation that the projection of a plane-segment equals the given plane-segment multiplied by the cosine of the projection angle.

127. An important special case of the theorem of § 126 is the area of the figure obtained by projecting a circle upon a plane not parallel to the plane of the circle.

Definition: The figure obtained by projecting a circle upon a plane not parallel to the plane of the circle, nor at right angles to it is called an **ellipse**.

128. Area of the Ellipse. — In the figure two planes, M and M', meet in a line PQ. The circle O in M has a diameter $AB \parallel PQ$ and a diameter $CD \perp PQ$.

In projecting the whole figure upon the plane M' the diameter AB projects into its equal $A'B'$, while CD projects into $C'D'$ so that $C'D' = CD \cos \angle 1$.



By theorem § 126 the area of the ellipse equals the area of the circle multiplied by $\cos \angle 1$.

Hence, $\pi r^2 \cdot \cos \angle 1$ is the area of the ellipse.

But $r \cos \angle 1 = O'C'$ and $r = O'B'$. (§ 120)

Hence, the area of the ellipse is $\pi \cdot O'C' \times O'B'$.

The segments $A'B'$ and $C'D'$ are called respectively the major and minor axes of the ellipse, and $O'B'$ and $O'C'$ the semimajor and the semiminor axes. These latter are usually denoted by a and b.

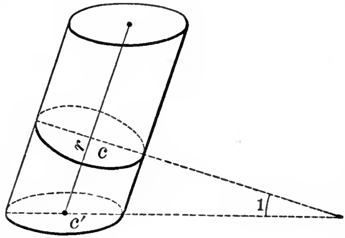
Hence, the area of the ellipse is πab.

Note that when a and b are equal, the ellipse becomes a circle, and this formula reduces to πa^2 as it should.

It may be of interest to note that the problem of finding the *length* of the ellipse is very much more difficult, and can be solved only by means of higher mathematics.

129. The figure of § 128 may also be regarded as representing a cylindrical surface of which the ellipse with center O' is a right section.

It is also true that if we start with a circular cylinder, that is, one whose right section c is a circle, then every oblique section of it, as c', is an ellipse.



The minor axis of such an ellipse will be the diameter $2r$ of the circle, and the major axis $2r \div \cos \angle 1$.

Thus, if a circular cylinder of diameter 6 in. is cut by a plane making an angle of 30° with the right section c, then the section c' thus made is an ellipse whose axes are 6 in. and $6 \div \cos 30° = 6 \div .866 = 6.93$ in. nearly.

Hence, the area of this elliptical section is

$$\pi ab = 3.14 \times 6.93 \times 6 = 130.56 \text{ sq. in.}$$

SUMMARY OF CHAPTER II.

1. Make a list of the definitions on prisms and also a list of those on cylinders and compare them.

2. Make a list of the theorems on prisms and also of those on cylinders and compare them.

3. State the axioms of this chapter and note that they all refer to areas and volumes which involve *curved* surfaces. Compare these with the axioms on the circle in Plane Geometry.

4. Make a list of all the formulas given by the theorems and corollaries of this chapter.

5. Make an outline of the definitions and theorems concerning projections of lines and surfaces.

6. Explain how the area of an ellipse is obtained either by projecting it into a circle or by projecting a circle into it.

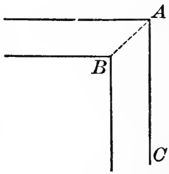
7. Make a list of the applications in this chapter which have appealed to you as interesting or practical or both. Return to this question, after studying the problems and applications which follow.

PROBLEMS AND APPLICATIONS.

1. Given a right circular cylinder the radius of whose base is 6 inches. Find the area of an oblique cross section inclined at an angle of 45° to the plane of the base.

2. Given an oblique circular cylinder the radius of whose right section is 10 inches. Find the area of the base if it is inclined at an angle of 60° to the right section.

3. If an oblique circular cylinder has an altitude h, an element e, radius of right section r, and $\angle A$ the inclination of the base to the right section, express the volume in two ways and show that these are equivalent.



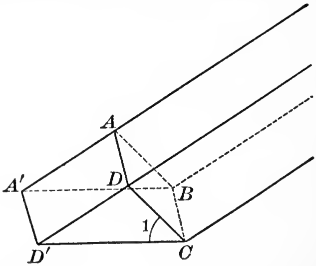
4. A six-inch stovepipe has a 45° elbow, that is, it turns at right angles. (The angle CAB is called the elbow angle.) Find the area of the cross section at AB. Likewise if it has a 60° elbow angle.

5. At what angle must the damper in a circular stovepipe be turned in order to obstruct just half the right cross sectional area of the pipe?

SUGGESTION. The damper must be turned through an angle such that the area of the projection of the damper upon a right cross section is equal to half that cross section.

6. The comparatively low temperature of the earth's surface near the pole, even in summer, when the sun does not set for months, is due largely to the *obliqueness* with which the sun's rays strike the earth. That is, a given amount of sunlight is spread over a larger area than in lower latitudes.

Thus, if in the figure $D'C$ is a horizontal line, and $D'D$ the direction of the sun's rays, then a beam of light whose right cross section is $ABCD$ is spread over the rectangle $A'BCD'$. In other words, a patch of ground $A'BCD'$ receives only as much sunlight as a patch the size of $ABCD$ receives when the sun's rays strike it vertically.



Area $ABCD =$ area $A'BCD' \cos\angle 1$,

or

$$\text{area } A'BCD' = \text{area } ABCD \times \frac{1}{\cos\angle 1}.$$

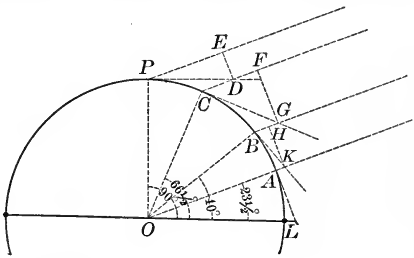
Hence, each unit of area in $A'BCD'$ receives $\cos \angle 1$ times as much light as a unit in $ABCD$.

Hence, to compare the heat-producing powers of sunlight in any latitude with that at the place where the sun's rays fall vertically, we need to know how the projection angle, $\angle 1$, is related to the difference in latitude of the two places.

7. If $\angle 1 = 30°$, compare the amount of heat received by a unit of area in $ABCD$ and $A'BCD'$.

8. What must $\angle 1$ be in order that a unit of area in $A'BCD'$ shall receive only $\frac{1}{5}$ as much light as a unit in $ABCD$?

9. The figure represents a cross section of the earth with an indication of the direction of the rays of light as they strike it at the summer solstice when they are vertical at A, the tropic of Cancer. B represents the latitude of Chicago, C the polar circle, and P the north pole. The angles PDE, CGF, BKH represent the projection angle, $\angle 1$, for the various latitudes.



Prove that $\angle PDE = \angle POK$, $\angle CGF = \angle COK$, $\angle BKH = \angle BOK$.

That is, $\angle 1$ for each place is the latitude of that place minus the latitude of the place where the sun's rays are vertical.

10. Find the relative amount of sunlight received by a unit of area at the tropic of Cancer and at the north pole at the time of the summer solstice.

SUGGESTION. The required ratio is $\dfrac{1}{\cos \angle 1} = \dfrac{1}{\cos .66\frac{1}{2}°} = \dfrac{1}{.399}$.

11. Find the ratio between the amount of light received by a unit of the earth's area at Chicago and at the tropic of Cancer at the time of the summer solstice.

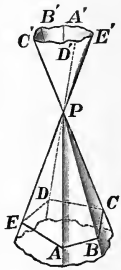
12. Find the same ratio for the polar circle and the tropic of Cancer at the spring equinox, when the sun is vertical over the equator.

13. Find the same ratio for the equator and Chicago at the winter solstice when the sun is vertical at latitude $23\frac{1}{2}°$ south.

CHAPTER III.

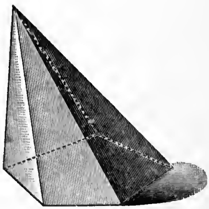
PYRAMIDS AND CONES.

130. Definitions. Given a convex polygon and a fixed point not in its plane. If a line through the fixed point moves so as always to touch the polygon and is made to traverse it completely, the line is said to generate a convex **pyramidal surface.**



The moving line is the **generator** of the surface, and in any of its positions it is an **element** of the surface. The guiding polygon is the **directrix**, and the fixed point the **vertex.** A pyramidal surface has two parts, called **nappes**, on opposite sides of the fixed point. Compare definitions in § 148.

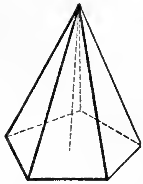
A polyhedral angle is a pyramidal surface of one nappe. See § 63.



131. That part of a pyramidal surface included between the fixed point and a plane cutting all its elements, together with the intercepted segment of the plane, is called a **pyramid.**

The intercepted plane-segment is the **base** of the pyramid, and the part of the pyramidal surface between the base and the vertex is the **lateral surface.**

The lateral surface is composed of **triangles** having a common vertex at the vertex of the pyramid, and having as bases the sides of the polygonal base. The sides common to two such triangles are the **edges** of the pyramid.



Pyramids are classified according to the shape of the base, as **triangular, quadrangular, pentagonal,** etc.

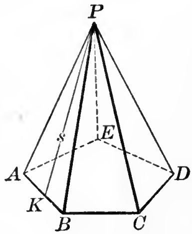
A pyramid having a triangular base has, in all, four faces, and is called a **tetrahedron.** In this case every face is a triangle, and any one may be taken as the base.

The **altitude of a pyramid** is the perpendicular distance from the vertex to the base.

A **regular right pyramid**, or simply a **regular pyramid**, is one whose base is a regular polygon such that the perpendicular from the vertex upon it meets it at the center.

EXERCISE. Show that the faces of a regular right pyramid are congruent isosceles triangles, and hence have equal altitudes.

The **slant height** of a regular right pyramid is the altitude of any one of its triangular faces.



132. THEOREM. *The lateral area of a regular right pyramid is equal to one half the product of its slant height and the perimeter of the base.*

Suggestion. Calling L the lateral area, $s = KP$ the slant height, and $p = AB + BC + CD + DE + EA$, the perimeter, show that

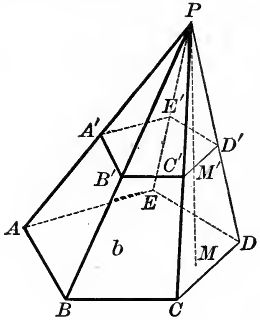
$$L = \tfrac{1}{2}\, sp.$$

133. THEOREM. *If a pyramid is cut by a plane parallel to the base:*

(1) *The edges and the altitude are divided in the same ratio.*

(2) *The polygonal section is similar to the base.*

(3) *The areas of this section and of the base are in the same ratio as the squares of their perpendicular distances from the vertex.*



Outline of proof: Given $ABCDE \parallel A'B'C'D'E'$.

(1) To prove that $\frac{PM'}{PM} = \frac{PA'}{PA} = \frac{PB'}{PB}$, etc., pass another plane through P parallel to the base and then use § 37.

(2) To prove that $ABCDE \sim A'B'C'D'E'$, we show that $\angle A = \angle A'$, $\angle B = \angle B'$, etc., and also $\frac{A'B'}{AB} = \frac{B'C'}{BC}$, etc.

(3) Calling the area of the cross section b' and that of the base b, we are to prove that $\frac{b'}{b} = \frac{\overline{PM'}^2}{\overline{PM}^2}$, and for this we need to show that $\frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{PM'}{PM}$.

Give all the steps in detail.

134. COROLLARY. *If two pyramids have equal altitudes and bases of equal areas lying in the same plane, the sections made by a plane parallel to the plane of the bases have equal areas.*

Suggestion. If t and t' are the areas of the sections and b and b' those of the bases, show by the theorem that $\frac{t}{b} = \frac{t'}{b'}$, and hence that $t = t'$ if $b = b'$.

135. **EXERCISES.**

1. What is the slant height of a regular right pyramid if its lateral area is 160 sq. in. and the perimeter of its base is 20 in.?

2. What is the perimeter of the base of a regular right pyramid if its lateral area is 250 sq. in. and its slant height is 17 in.?

3. How could you find the lateral area of a *regular* right pyramid? Of any irregular pyramid? What measurements would be necessary in each case? Why are fewer measurements needed in the case of a regular right pyramid?

4. The base of a regular right pyramid is a regular hexagon whose side is 8 ft. Find the lateral area if the altitude of the pyramid is 6 ft.

5. The lateral area of a regular right hexagonal pyramid is 48 sq. ft. and the slant height is 12 ft. Find the altitude of the pyramid.

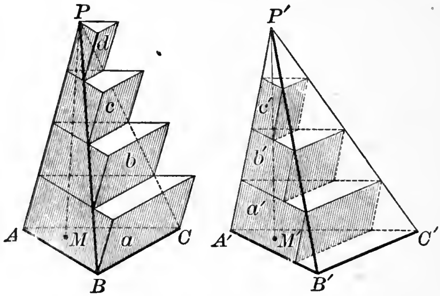
6. The base of a regular right pyramid is a square whose side is 16 ft., and the altitude of the pyramid is 6 ft. Find the lateral area.

7. A pyramid with altitude 8 and a base whose area is 36 is cut by a plane parallel to the base so that the area of the section is 18 sq. in. Find the distance from the base to the cutting plane.

8. If the altitude of a pyramid is h, how far from the base must a plane parallel to it be drawn so that the area of its cross section shall be half that of the base of the pyramid?

9. In a regular right pyramid a plane parallel to the base cuts it so as to make a section whose area is one half that of the base. Find the ratio between the lateral area of the pyramid and that of the small pyramid cut off by the plane.

136. Definition. A triangular pyramid is cut by a series of planes parallel to the base, including one through the vertex and also the one in which the base lies.



Through the lines of intersection of these planes with one of the faces, planes are constructed parallel to the opposite edge, thus forming a set of prisms all lying within the pyramid, as a', b', c', in pyramid P', or a set lying partly outside the pyramid, as a, b, c, d in pyramid P.

The inner prisms thus constructed are called **a set of inscribed prisms,** and the other **a set of circumscribed prisms.**

137. Axiom VII. *A pyramid has a definite volume which is less than the combined volume of any set of circumscribed prisms and greater than that of any set of inscribed prisms.*

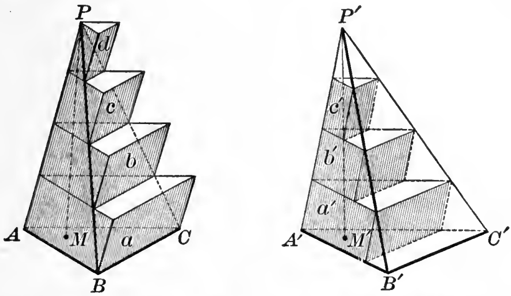
138. EXERCISES.

1. In the figure above prove that prisms d and c' are equal in volume. Also that $c = b'$, etc.

2. If the area of the base of pyramid P is 12 sq. in., what is the altitude of prism a if its volume is 1 cu. in.? If its volume is $\frac{1}{10}$ cu. in.?

3. If the altitude of the pyramid in the preceding exercise is 16 in., into at least how many equal parts must it be divided if the volume of prism a is to be less than 1 cu. in.? less than .01 cu. in.? Is it possible to divide the altitude of the pyramid into a sufficiently large number of equal parts to make the volume of prism a as small as we like?

139. THEOREM. *If two triangular pyramids have equal altitudes and bases of equal areas, their volumes are equal.*



pyramids

Given the ~~prisms~~ P and P', in which $PM = P'M'$, and the bases ABC and $A'B'C'$ have equal areas.

To prove that P and P' have equal volumes.

Proof: Divide PM and $P'M'$ into the same number of equal parts, and using these division points, construct a set of circumscribed prisms for P and a set of inscribed prisms for P'.

Then $\qquad a' = b,\ b' = c,\ c' = d.$ (See § 94.)

Denote $a + b + c + d$ by V and $a' + b' + c'$ by V'.

Then $\qquad V - V' = a.$ (Why?) (1)

If P differs at all in volume from P', let P be the greater, and let the difference be some fixed number, K, so that

$$P - P' = K. \qquad (2)$$

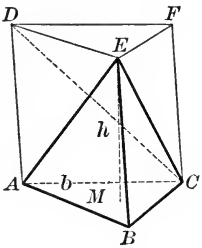
But from (1) $P - P' < a$, since $P < V$ and $P' > V'$.

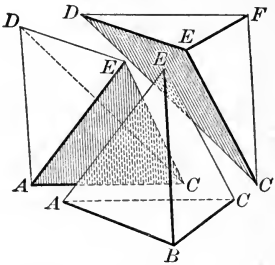
Now a can be made less than K by taking the divisions on PM small enough.

Hence, $\qquad P - P' < K.$ (3)

Thus (3) contradicts (2), and hence the supposition that P and P' differ in volume is impossible.

140. THEOREM. *The volume of a triangular pyramid is one third of the product of its base and altitude.*





Given the triangular pyramid *E–ABC*. Let *h*, *b*, and *V* be the numerical measures respectively of the altitude *EM*, the base *ABC* and the volume.

To prove that $V = \frac{1}{3}bh.$

Proof: Construct on the base *ABC* a triangular prism with altitude h and lateral edge *EB*.

This prism may be cut into three pyramids, as shown in the figure to the right, by the plane sections through *DEC* and *AEC*. See Note, § 68.

The pyramids *E–ABC* and *C–DEF* have the same volume (§ 139).

Likewise the pyramids *E–ACD* and *E–CFD* have the same volume (§ 139).

But *C–DEF* and *E–CFD* are only different notations for the same pyramid.

Hence, $E\text{–}ABC = C\text{–}DEF = E\text{–}ACD.$

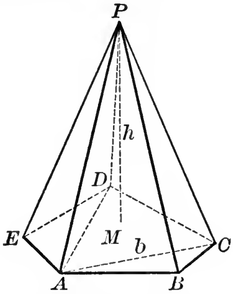
That is, *E–ABC* is one third of the prism.

But the volume of prism $= bh.$ (Why?)

Hence, $V =$ volume of pyramid $= \frac{1}{3}bh.$

State in detail the reasons for each step.

141. THEOREM. *The volume of any pyramid is one third of the product of its base and altitude.*



Given the pyramid *P–ABCDE*. Let *V*, *b*, and *h* be the numerical measures respectively of the volume, base, and altitude.

To prove that $V = \frac{1}{3}bh$.

Proof: By means of the diagonal planes *PAC* and *PAD*, divide the given pyramid into three triangular pyramids.

Complete the proof.

142. COROLLARIES. 1. *The volumes of any two pyramids having the same or equal altitudes are in the same ratio as the areas of their bases.*

2. *The volumes of any two pyramids having the same or equal bases are in the same ratio as their altitudes.*

143. EXERCISES.

1. The altitude of a certain pyramid is 14 in. and its volume is 380 cu. in. Find the area of its base.

2. The area of the base of a pyramid is 48 sq. ft. and its volume 260 cu. ft. Find its altitude.

3. Find the locus of the vertices of pyramids having the same base and equal volumes.

4. A diagonal of the square base of a regular right pyramid is $7\sqrt{2}$ in. and its volume 147 cu. in. Find its altitude and lateral area.

5. A flower bed is in the form of a regular right pyramid, with a square base 5 ft. on a side. The altitude is 2 ft. Find the number of cubic feet of soil in its construction.

6. A tent is to be made in the form of a right pyramid, with a regular hexagonal base. If the altitude is fixed at 15 ft., what must be the side of the base in order that the tent may inclose 350 cu. ft. of space?

7. Two marble ornaments of equal altitudes are pyramidal in form. One has a square base 2 in. on a side and the other a regular hexagonal base 1 in. on a side. Compare their volumes.

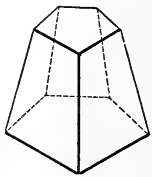
8. Two monuments having bases of equal areas are pyramidal in shape, one being 15 ft. high and the other 18 ft. Compare their volumes.

9. If the base and the volume of a pyramid are known, is it possible to determine its lateral area?

10. Given a pyramid with rectangular base. By how much is its volume multiplied if the length and width of the base and also the altitude are each multiplied by 2; by 3; by any number n?

11. Given a pyramid with altitude 10 and a regular hexagonal base, each of whose sides is 5. By how much is its volume multiplied if each side of the base and also the altitude is multiplied by 2; by 3; by 4; by any number n?

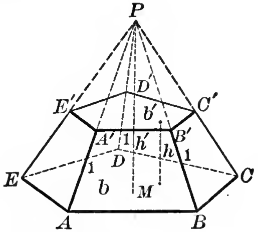
For a general statement of the law exemplified in these exercises, see § 195.



144. Definitions. The figure formed by the base of a pyramid, any cross section, and the portion of the lateral faces included between these planes, is called a **truncated pyramid.** If the cross section is parallel to the base, the figure is called a **frustum of a pyramid,** and this section is the **upper base.**

The **altitude** of a frustum is the perpendicular distance between its bases. The **slant height** of the frustum of a regular right pyramid is the common altitude of its trapezoidal faces.

145. THEOREM. *The volume of a frustum of a pyramid is equal to the combined volume of three pyramids whose common altitude is the same as that of the frustum, and whose bases are the upper and lower bases of the frustum and the mean proportional between these bases.*



Given the frustum AC' with lower base b, upper base b', and altitude h. Let h' be the altitude PM of the completed pyramid $P\text{–}ABCDE$. Then $\sqrt{bb'}$ is the mean proportional between b and b'.

To prove that the volume of AC' is

$$V = \tfrac{1}{3}\, h[b + b' + \sqrt{bb'}].$$

Proof: The altitude of the pyramid $P\text{–}A'B'C'D'E'$ is $h' - h$.

Hence, $$\frac{b}{b'} = \frac{h'^2}{(h'-h)^2}, \qquad (\S\ 133)$$

from which $$h' = \frac{h\sqrt{b}}{\sqrt{b} - \sqrt{b'}}. \qquad (1)$$

Now V is the difference between the pyramids whose altitudes are h' and $h' - h$.

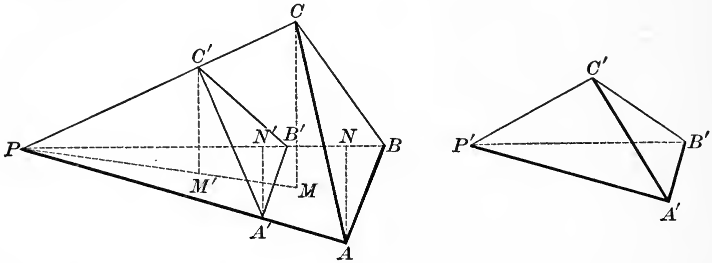
Hence, $$V = \tfrac{1}{3}\, bh' - \tfrac{1}{3}\, b'(h' - h),$$

or, rearranging, $$V = \tfrac{1}{3}\, b'h + \tfrac{1}{3}\, h'(b - b'). \qquad (2)$$

Substituting (1) in (2), $$V = \tfrac{1}{3}\, h[b + b' + \sqrt{bb'}].$$

Show all the details.

146. THEOREM. *The volumes of two tetrahedrons, having a trihedral angle of the one congruent to a trihedral angle of the other, are in the same ratio as the products of the edges which meet in the vertices of these angles.*



Given the tetrahedrons $P\text{–}ABC$ and $P'\text{–}A'B'C'$ whose volumes are V and V' and in which Tri. $\angle P =$ Tri. $\angle P'$.

To prove that $\dfrac{V}{V'} = \dfrac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}$.

Proof: Place $P'\text{–}A'B'C'$ so that Tri. $\angle P'$ coincides with Tri. $\angle P$.

Let CM and $C'M'$ be the altitudes of $P\text{–}ABC$ and $P\text{–}A'B'C'$ from the vertices C and C' upon the plane PAB.

Let AN and $A'N'$ be the altitudes of the △ PAB and $PA'B'$.

Then $\dfrac{V}{V'} = \dfrac{\frac{1}{3}\,CM \cdot \text{area } PAB}{\frac{1}{3}\,C'M' \cdot \text{area } PA'B'} = \dfrac{CM \cdot PB \cdot AN}{C'M' \cdot PB' \cdot A'N'}$. (1)

Now prove $\dfrac{CM}{C'M'} = \dfrac{PC}{PC'}$ and $\dfrac{AN}{A'N'} = \dfrac{PA}{PA'}$.

Hence, substituting in (1),

we have $\dfrac{V}{V'} = \dfrac{PC \cdot PB \cdot PA}{P'C' \cdot P'B' \cdot P'A'}$.

Give all the steps and reasons in detail.

147. **EXERCISES.**

1. Show that the lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids. Hence find the area of its lateral surface in terms of the slant height and the perimeters of the bases.

2. Show that sections of a pyramid made by two planes parallel to the base are similar polygons whose areas are in the same ratio as the squares of the distances from the vertex.

3. Show that any two pyramids standing on the same base, or on equal bases in the same plane, have the same volume if their vertices coincide or lie in a plane parallel to the base.

4. Show that the volumes of two pyramids have the same ratio as the areas of their bases if they have equal altitudes, and the same ratio as their altitudes if they have equal bases.

5. A frustum of a pyramid is cut from a pyramid the perimeter of whose base is 60 inches and whose altitude is 15 inches. What is the altitude of the frustum, if the perimeter of its upper base is 40 inches?

Does the result depend upon the number of sides of the pyramid?

6. Solve the preceding problem if the perimeter of the upper base of the frustum is one nth that of the lower base. Does this result depend upon the number of sides of the pyramid?

7. The area of the base of a pyramid is 180 square inches and its altitude is 20 inches. Cut from it a frustum, the area of whose upper base is 45 square inches; also one the area of whose upper base is one nth of 180 square inches. Do these results depend upon the number of sides of the pyramid?

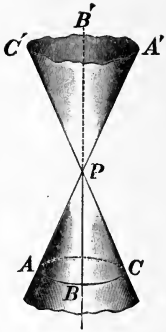
8. Two triangular pyramids have equal trihedral angles at the vertex. The lateral edges of one pyramid are 14, 16, and 18, and those of the other 7, 8, and 9. Find the ratio between their volumes. Are the data given sufficient to find the volume of each pyramid?

9. If two triangular pyramids have equal trihedral angles at the vertex and if the lateral edges of one are a, b, and c, and two lateral edges of the others are a' and b', find the third lateral edge of the second pyramid so that their volumes shall be equal.

10. The slant height of a frustum of a regular pyramid is 10 inches and the apothems of its bases 8 and 6 inches respectively. Find its altitude.

CONES.

148. Definition. Given a closed convex curve and a fixed point not in its plane. If a line through the fixed point moves so as always to touch the curve and is made to traverse it completely, it is said to generate a **convex conical surface.**

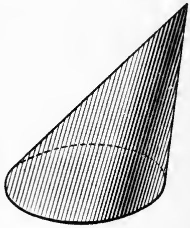


The moving line is called the **generator** of the surface, and in any particular position it is an **element** of the surface.

The fixed curve is called the **directrix,** and the fixed point the **vertex.**

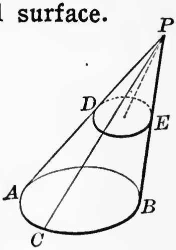
A conical surface consists of two parts, called **nappes,** on opposite sides of the fixed point.

149. That part of a convex conical surface included between its vertex and a plane cutting all its elements, together with the intercepted portion of the plane, is called a **cone.**



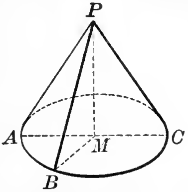
The intercepted part of the plane is the **base** of the cone, and the curved surface is its **lateral surface.**

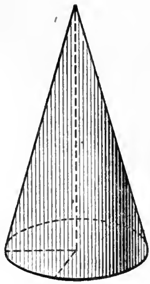
The **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.



A **circular** cone is one which has a circular cross section such that the perpendicular upon it from the vertex meets it at the center. If the base is such a circle, the cone is then called a **right circular cone.** Otherwise, it is an **oblique circular cone.**

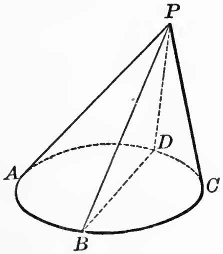
A right circular cone may be generated by rotating a right triangle *PMB* about one of its legs, *PM*, as an axis. The hypotenuse *PB* generates the conical surface, and the other leg, *MB*, generates the base.





The generator of a right circular cone in any position is called the slant height.

150. THEOREM. *If a plane contains an element of a cone and meets it in one other point, then it contains another element also, and the section is a triangle.*



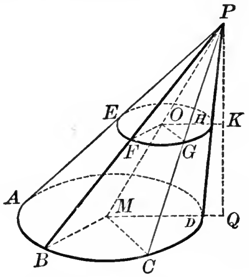
Let a plane contain the element *PD* of the cone *P–ABC*, and also one other point *B*.

To prove that this plane contains another element *PB*, and that the section is a triangle *PBD*.

Suggestion. Connect *P* and *B*. This segment lies in the conical surface. (Why?) Complete the proof by showing that *BD* lies in the base of the cone. Compare this proof with that of § 100.

151. Definition. If a plane contains an element of a cone and no other point of the cone, the plane is **tangent** to the cone, and the element is called the **element of contact.**

152. THEOREM. *If the base of a cone is circular, every plane section parallel to the base is also circular.*



Given a cone with a circular base AD.

To prove that the $\parallel$ section EH is also circular.

Proof: Draw the straight line from P to the center M of the base, and let it meet the section EH in the point O. Let F and G be any two points on the perimeter of the section EH.

Pass planes containing PM through the points F and G, and let them cut the base in MB and MC respectively.

Now in the $\triangle$s PMB and PMC prove that $OF = OG$.

Hence, as F and G, *any* two points on the perimeter of this section, are equally distant from O, this shows that EH is a circle whose center is O.

153. COROLLARY. *If a cone has a circular base, the areas of two parallel cross sections are in the same ratio as the squares of their perpendicular distances from the vertex and also as the squares of the distances of their centers from the vertex.*

Suggestion. Use the figure of § 152, and let PQ be the altitude of the cone. Then show that

$$\frac{\text{Area } AD}{\text{Area } EH} = \frac{\overline{MD}^2}{\overline{OH}^2} = \frac{\overline{PM}^2}{\overline{PO}^2} = \frac{\overline{PQ}^2}{\overline{PK}^2}.$$

154. EXERCISES.

1. Into how many parts do the two nappes of a conical surface or of a pyramidal surface divide the remaining points of space?

2. If in constructing a conical surface a polygon is used as a directrix instead of a closed convex curve, what kind of surface is obtained?

3. Why is it specified in the definition of the convex conical surface that the vertex must not lie in the same plane as the directrix?

4. How many cones may be cut from a conical surface if they are to have no point in common except the vertex?

5. If a triangle which is not a right triangle is made to revolve about one of its sides, does it generate a cone?

6. Can every circular cone be developed by revolving a right angled triangle about one of its sides?

7. If a cone has a circular base, the line from the vertex to the center passes through the center of every plane section parallel to the base.

8. If a cone has a circular base, the plane determined by a tangent to the base and the element at the point of tangency is a tangent plane to the cone.

9. Through a point outside a cone with a circular base, how many planes are there which are tangent to the cone?

10. The diameter of the circular base of a cone is 8 in. and the altitude of the cone 9 in. A plane parallel to the base cuts the cone in a section whose diameter is 3 in. Find the distance from the vertex of the cone to this plane.

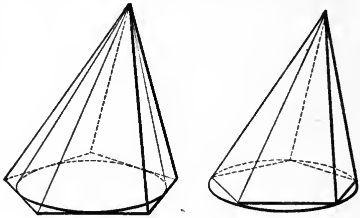
11. If the area of the circular base of a cone is 16π sq. in. and its altitude 6 in., find the distance from the vertex to a plane, parallel to the base, which cuts the cone in a section with area 9π sq. in.

12. The area of the circular base of a cone is b sq. in. and its altitude h in. Find the distance from the base to a plane parallel to it, which cuts off a cone the area of whose base is one nth that of the base of the original cone.

13. Compare the exercises on the cone thus far studied with those on the pyramid given on pages 76, 79. Note that a pyramid can be made to approximate very closely to a cone by making its number of faces very large.

MEASUREMENT OF THE SURFACE AND VOLUME OF A CONE.

155. **Definitions.** A pyramid is said to be **inscribed in a cone** if its lateral edges are elements of the cone, and the bases of the cone and the pyramid lie in the same plane, as in Fig. 2.



(1) (2)

A pyramid is said to be **circumscribed about a cone** if its lateral faces are all tangent to the cone, and the bases of the cone and the pyramid lie in the same plane, as in Fig. 1.

156. THEOREM. *In a right circular cone a pyramid may be inscribed whose slant height differs from the slant height of the cone by less than any given fixed number.*

Proof: Let d be the given fixed number. From plane geometry we know that a regular polygon may be inscribed in the base of the cone whose apothem differs from the radius of the base by less than d. Let this regular polygon be the base of an inscribed pyramid. Then the slant height of this pyramid differs from that of the cone by less than d, since the difference of two sides of a triangle is less than the third side.

157. **Axiom VIII.** *The lateral surface of a convex cone has a definite area, and the cone incloses a definite volume, which are less respectively than those of any circumscribed pyramid and greater than those of any inscribed pyramid.*

158. THEOREM. *The area of the lateral surface of a right circular cone is equal to one half the product of its slant height and the circumference of its base.*

Given a right circular cone of which s is the slant height, c is the circumference of the base, and L the lateral area.

To prove that $L = \frac{1}{2} sc.$

Proof: Suppose that $L > \frac{1}{2} sc.$ Then $L = \frac{1}{2} sK$ (1)

where $K > c.$

Circumscribe about the cone a pyramid the perimeter of whose base is p, such that $p < K$. (Why is this possible?)

Hence, $\frac{1}{2} sp < \frac{1}{2} sK.$ (2)

That is, (2) contradicts (1) because of § 157.

Next suppose $L < \frac{1}{2} sc.$ Then $L = \frac{1}{2} sK'$ (3)

where $K' < c.$

Let $c - K' = d$. By § 352, Plane Geometry, a polygon may be inscribed in the base of the cone whose perimeter p differs from c by as little as we please, and by § 156 a pyramid may be inscribed in the cone whose slant height s' differs from the slant height s of the cone by as little as we please. Hence a pyramid may be inscribed such that $\frac{1}{2} s'p$ differs from $\frac{1}{2} sc$ by less than d.

That is $\frac{1}{2} s'p > \frac{1}{2} sK.$ (4)

But (4) contradicts (3) because of § 157.

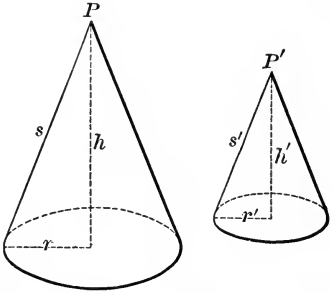
Since therefore L is neither less than nor greater than $\frac{1}{2} sc$, it must be equal to $\frac{1}{2} sc$.

159. COROLLARY. *If r is the radius of the base of a right circular cone and s the slant height, then*

$$L = \frac{1}{2} \cdot 2\,\pi r S = \pi r S.$$

160. Definition. Two right circular cones are **similar** if they are generated by two similar right triangles revolving about corresponding sides.

161. THEOREM. *The lateral areas or the entire areas of two similar right circular cones are in the same ratio as the squares of their altitudes, their slant heights, or the radii of their bases.*



Given the cones P and P', with altitudes h, h', slant heights s, s', radii of bases r, r', lateral areas L, L', and entire areas A, A'.

To prove that

$$\frac{A}{A'} = \frac{L}{L'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}.$$

Proof: See the suggestions under § 111. Give all the steps.

162. EXERCISES.

1. The lateral area of a cone is 36 square inches. What is the lateral area of a similar cone whose altitude is $\frac{3}{4}$ that of the given cone?

2. The total area of one of two similar cones is three times that of the other. Compare their altitudes and also their radii.

3. The sum of the total areas of two similar cones is 144 square inches. Find the area of each cone if one is $1\frac{3}{4}$ times as high as the other.

4. Prove that if in two tetrahedrons three faces of one are congruent respectively to three faces of the other and similarly placed about a vertex, the tetrahedrons are congruent.

5. Prove that if in two tetrahedrons two faces and the included dihedral angle are congruent and similarly placed, the tetrahedrons are congruent.

6. A pedestal for a monument is in the shape of a frustum of a regular hexagonal pyramid, the radius of the upper base being 4 ft., that of the lower base 6 ft., and the altitude of the frustum 8 ft. Find its volume, slant height, and lateral surface.

7. Find the volume of a frustum of a pyramid the areas of whose bases are 25 sq. in. and 18 sq. in. and and whose altitude is 6 in.

8. The area of the lower base of a frustum is 42 sq. ft., its altitude 8 ft., and volume 200 cu. ft. Find the area of the upper base.

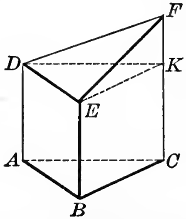
9. The area of the base of a pyramid is 480 sq. ft. and its altitude 30 ft. Find the volume of the frustum remaining after a pyramid with altitude 10 ft. has been cut off by a plane parallel to the base.

10. The area of the base of a pyramid is 250 sq. in. If a plane section of the pyramid parallel to the base and at a distance of 5 in. from it has an area of 175 sq. in., find the altitude of the pyramid.

11. The sides of the base of a triangular pyramid are 6 ft., 8 ft., 10 ft., and its volume 96 cu. ft. Find its altitude.

12. What part of the volume of a cube is a frustum of a pyramid cut from a pyramid whose base is one face of the cube and whose vertex lies in the opposite face, if the altitude of the frustum is one half the edge of the cube?

13. Find the dihedral angle at the base of a regular pyramid if the altitude is one half the slant height.

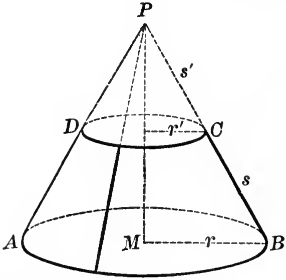


14. A right triangular prism is cut by a plane not parallel to the base, but such that its intersection DE is parallel to the base segment AB. Show that the volume of the part thus cut off is one third the product of the sum of the three vertical edges and the area of the base.

SUGGESTION. Draw plane $DEK \parallel ABC$.

15. Find the volume of a truncated triangular prism, the area of whose base is 25 square inches and whose lateral edges are 8, 7, 8.

163. THEOREM. *The lateral area of a frustum of a right circular cone is equal to one half of the sum of the circumferences of the bases multiplied by the slant height.*



Given the frustum $ABCD$, with slant height s and radii r and r'. Let L represent its lateral area.

To prove that $L = \frac{1}{2}(2\pi r + 2\pi r')s = \pi s(r + r')$.

Proof: Complete the cone, and let $PC = s'$.

Then $$L = \tfrac{1}{2}[2\pi r(s + s') - 2\pi r's']$$
$$= \pi rs + \pi s'(r - r'). \qquad (1)$$

But $$\frac{r}{r'} = \frac{s + s'}{s'}, \text{ from which } s' = \frac{r's}{r - r'}. \qquad (2)$$

Substituting s' from (2) in (1),
$$L = \pi rs + \pi r's = \pi s(r + r').$$

164. COROLLARY. *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases multiplied by the slant height.*

Suggestion. From the theorem
$$L = \pi s(r + r') = 2\pi \frac{(r + r')}{2} s.$$

Now show that $\frac{r + r'}{2}$ is the radius of the section midway between the two bases.

165. EXERCISES.

1. The lateral surface of a right circular cone is 75 sq. ft. Find the altitude if the radius of the base is 4 ft.

2. A circular chimney 100 ft. high is in the form of a frustum of a right cone whose lower base is 10 ft. in diameter and upper base 8 ft. Find the lateral surface.

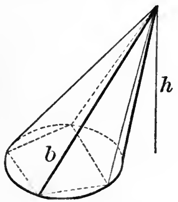
3. The lateral area of a frustum of a right circular cone is 60π sq. in., the radii of the two bases are 6 in. and 4 in. Find the slant height of the frustum.

4. Find the altitude of the frustum in the preceding example, and also the altitude of the cone from which it is cut.

5. A frustum of a right circular cone has an altitude one half that of the cone from which it is cut. If its slant height is 8 ft. and lateral area 64π sq. ft., find the diameters of its bases.

6. Find the altitude of the frustum of the cone in the preceding example; also the lateral area of the cone from which it was cut.

166. THEOREM. *The volume of any convex cone is equal to one third the product of its base and altitude.*

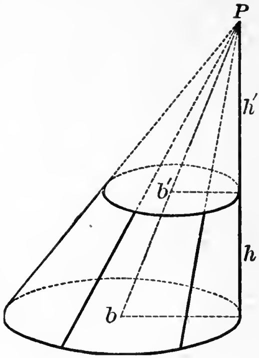


Suggestion. Let h be the altitude, b the area of the base, and V the volume.

Show that V cannot be different from $\frac{1}{3}bh$ by an argument similar to that of § 108, making use of § 141.

167. COROLLARY. *If a cone has a circular base of radius r and altitude h, then* $V = \frac{1}{3}\pi r^2 h$.

168. THEOREM. *The volume of the frustum of a convex cone is equal to the combined volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the upper and lower bases of the frustum and a mean proportional between these bases.*



Suggestion. The proof is exactly like that of § 145.

NOTE. Observe that the two preceding theorems apply to *any* convex cone because the *altitude h is constant.* The actual computation is *practicable* only when the areas of the bases can be found, as in the case of the circle, or ellipse.

SUMMARY OF CHAPTER III.

1. Make a list of definitions on pyramids and also one on cones and compare them.

2. Make a list of theorems on pyramids and also one on cones and compare them.

3. What axioms have been used in this chapter? Compare these with the axioms in Chapter II.

4. Make a list of all the formulas given by the theorems of this chapter and compare them with the corresponding formulas in Chapter II.

5. What theorems on cylinders have no corresponding theorems for cones?

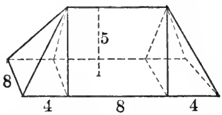
6. Show that a frustum of a cone becomes more and more nearly identical with a cylinder if the vertex of the cone is removed farther and farther from the base.

7. Make a list of the applications in this chapter which have impressed you as interesting or practical or both. Return to this question again after studying the problems and applications which follow.

PROBLEMS AND APPLICATIONS.

1. Show that the volume of a right triangular prism is equal to one half the product of the area of one face and the distance from the opposite edge to that face.

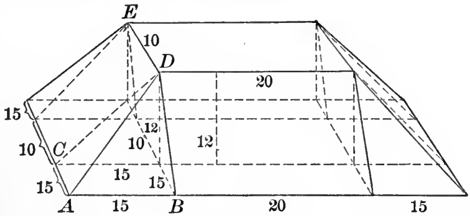
2. A mound of earth in the shape shown in the figure has a rectangular base 16 yards long and 8 yards wide. Its perpendicular height is 5 yards, and the length on top is 8 yards. Find the number of cubic yards of earth in the mound.



SUGGESTION. If from each end a pyramid with a base 8 yd. by 4 yd. is removed, the remaining part is a triangular prism.

3. Given a figure in general shape the same as the preceding, with a rectangular base of length 24 ft. and width 6 ft. Find its volume and lateral area if the dihedral angles around the base are each 45°.

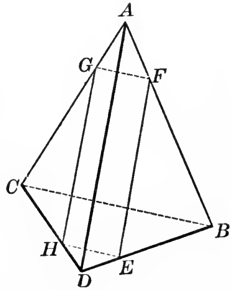
4. The accompanying figure represents a solid whose base is a



rectangle 50 units long and 40 units wide. Its height is 12 units and its top a rectangle 20 units by 10 units. Find its volume.

SUGGESTION. Divide the solid as indicated in the figure. Notice that this is not a frustum of a pyramid.

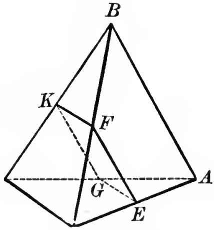
5. In a figure like the foregoing, how can we determine whether or not it represents the frustum of a pyramid?

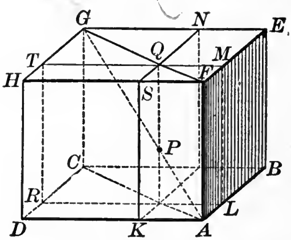


6. Show how to pass a plane through a tetrahedron so that the section shall be a parallelogram.

SUGGESTION. Pass a plane parallel to each of two opposite edges. See Ex. 6, page 15.

7. If the middle points of four edges of a tetrahedron, no three of which meet at the same vertex, are joined, a parallelogram is formed. Prove.





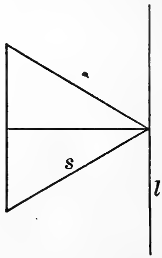
8. If through any point P in a diagonal of a parallelopiped planes KN and RM are drawn parallel to two faces, show that the parallelopipeds DQ and LN thus formed have equal volumes.

9. Find the volume and area of a figure formed by revolving an equilateral triangle about an altitude, the sides of the triangle being s.

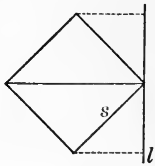
10. Find the area and volume of the figure developed by an equilateral triangle with sides s if it is revolved about one of its sides.

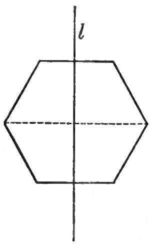
11. Find the area and volume of the figure developed by revolving a square whose side is s about one of its diagonals.

12. Through one vertex of an equilateral triangle with sides s draw a line l perpendicular to the altitude upon the opposite side. Find the volume and area of the figure developed by revolving the triangle about the line l.



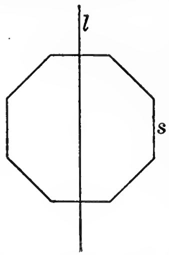
13. Through a vertex of a square with sides s draw a line l perpendicular to the diagonal through that vertex. Find the area and volume of the figure developed by turning the square around the line l.





14. In a regular hexagon with sides s draw a line l bisecting two opposite sides. Find the area and volume of the figure developed by turning the hexagon about l as an axis.

15. Solve a problem like the preceding, using a regular octagon instead of a hexagon.



16. If several planes are tangent to the same cone, find one point common to them all.

17. Find the locus of all lines which make a given angle with a given line at a given point in it.

18. Find the locus of all lines which make a given angle with a given plane at a given point.

19. One angle of a right triangle is 30°. Find the ratios between the surfaces of the solids developed by revolving this triangle around each of its three sides.

20. Find the ratios between the volumes of the solids developed in the preceding example.

21. Find the total area and the volume of a regular tetrahedron each of whose edges is e. (See § 170.)

22. If the numerical values of the volume and of the total area of a regular tetrahedron are equal, what is the length of its edge?

23. Find the length of an edge of a regular tetrahedron if its volume is numerically equal to the square of an edge.

24. Cut a pyramid of altitude h by means of a plane parallel to the base so that the perimeter of the section shall be half that of the base. Also cut it so that the perimeter of the section shall be $\frac{1}{3}$ that of the base.

25. Cut a right circular cone by 3 planes, each parallel to the base, so that the perimeters of the sections shall be $\frac{p}{4}, \frac{2p}{4}, \frac{3p}{4}$, p being the perimeter of the base. Find the distances from the vertex to the planes.

26. Cut a right circular cone of altitude h by a plane parallel to the base so that the area of the section shall be half that of the base. Find the distance from the vertex to the plane.

27. Show that the lateral area of the small cone cut off in the preceding example is one half the lateral area of the original cone.

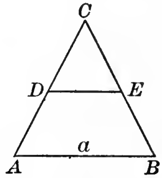
28. Cut a pyramid of altitude h by n planes, each parallel to the base, so that the areas of the sections shall be $\frac{A}{n+1}, \frac{2A}{n+1}, \frac{3A}{n+1}, \cdots,$

$\frac{(n-1)A}{n+1}$, A being the area of the base. Show that the distances from the vertex to the planes are $h\sqrt{\frac{1}{n+1}}$, $h\sqrt{\frac{2}{n+1}}$, $h\sqrt{\frac{3}{n+1}}$, ...

29. Cut a cone with altitude h by a plane parallel to the base so that the volume of the frustum formed shall equal half that of the cone. Find the distance from the vertex of the cone to the plane.

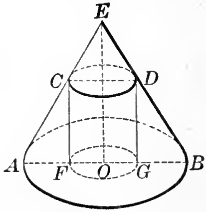
30. Cut a cone of altitude h by n planes, each parallel to the base, so that the frustums formed and the one ~~pyramid~~ Cone cut off at the top shall all have equal volumes.

31. An equilateral triangle ABC is swung around the line DE as an axis, D and E being middle points of the sides of the triangle. Find the volume of the figure thus developed by the trapezoid $ABED$ if $AB = a$.



32. Find the total surface of the figure in the preceding example.

33. In a right circular cone, with altitude h, and r the radius of its base, a cylinder is inscribed as shown in the figure. Find the radius OF of the cylinder if the area of the ring bounded by the circles OF and OA is equal to the lateral area of the small cone cut off by the upper base of the cylinder.



34. The same as the preceding, except that the lateral area of the small cone is to equal the lateral area of the cylinder.

35. Find the dihedral angles of a regular tetrahedron.

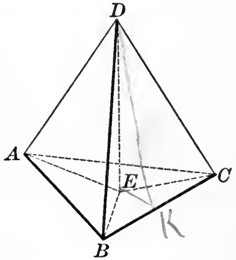
CHAPTER IV.

REGULAR AND SIMILAR POLYHEDRONS.

REGULAR POLYHEDRONS.

169. Definitions. A polyhedron is said to be **regular** if its polyhedral angles are all congruent and its faces are congruent regular polygons.

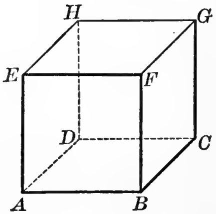
170. Construction of regular polyhedrons. Certain regular polyhedrons are very simple of construction, as indicated below.



(1) **The regular tetrahedron.** At the center E of an equilateral triangle ABC erect a perpendicular to the plane of the triangle. On this take a point D so that $AD = AC$.

Now prove that the four triangles, ABC, ACD, ABD, BCD, are *regular* and *congruent*, and that the four trihedral angles are congruent.

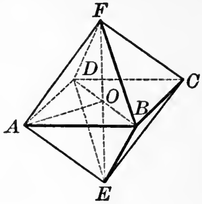
Suggestion. $AE = BE = CE$.



(2) **The regular hexahedron or cube.** At the vertices of a given square erect perpendiculars to its plane equal in length to the sides, and join their upper extremities as shown in the figure.

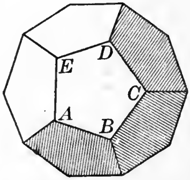
Show that six equal and congruent squares are formed and also eight congruent trihedral angles.

(3) **The regular octahedron.** Through the center O of a square $ABCD$ draw a perpendicular to the plane of the square.



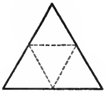
On this take points E and F such that $AF = AE = AB$. Join E and F to each of the four vertices, A, B, C, D.

Now prove that the eight faces are congruent regular triangles, and that the six polyhedral angles are congruent.

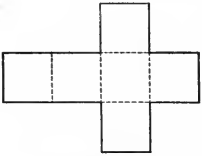


There are two other regular polyhedrons, namely :

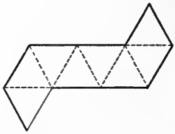
(4) **The regular dodecahedron,** having for faces twelve regular pentagons, and



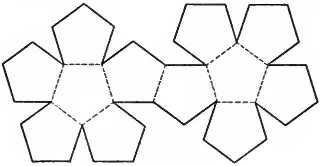
TETRAHEDRON



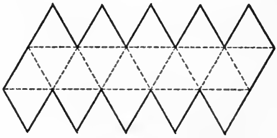
HEXAHEDRON



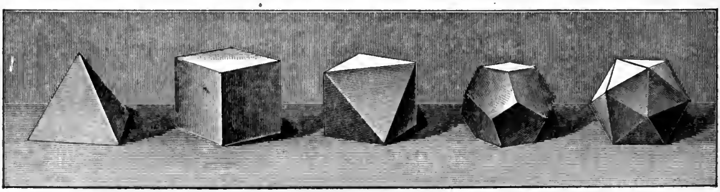
OCTAHEDRON

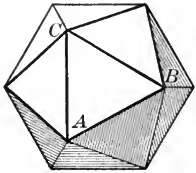


DODECAHEDRON



ICOSAHEDRON





(5) **The regular icosahedron,** having for faces twenty equilateral triangles.

The geometric constructions for (4) and (5) are not so simple as for the others. However, cardboard models of all five may be made by cutting out the figures, as shown herewith, and folding them along the dotted lines. They may be held in shape by means of gum paper stuck over the joints.

171. The number of regular polyhedrons. It will now be shown that there are *not more* than these five regular polyhedrons.

There must be *at least three* faces meeting at each vertex. If these are regular triangles, there may be *three*, as in the tetrahedron, or *four*, as in the octahedron, or *five*, as in the icosahedron; *but there cannot be six*, for in that case the sum of the angles about a vertex would be 360°, and it is readily seen that this sum cannot be as great as 360°. For a proof of this fact, see § 271.

If the faces are squares, there may be *three* about a vertex, as in the cube, *but there cannot be four*, for in that case the sum of the face angles at a vertex would be 360°.

If the faces are regular pentagons, there may be *three* about a vertex, making the sum of the face angles $3 \cdot 108° = 324°$, *but there cannot be four*, for then the sum would be greater than 360°. See § 271.

Regular polygons of more than five sides cannot form the faces of a regular polyhedron, for the sum of the face angles at a vertex would in any such case be more than 360°. Show why this is so.

Hence, there cannot be more regular convex polyhedrons than those exhibited above.

172. THEOREM. *If v is the number of vertices of a convex polyhedron, e its number of edges, and f its number of faces, then* $e + 2 = \overset{4}{v} + \overset{4}{f}$.

This is called Euler's Theorem. The proof is too difficult for an elementary text-book such as this. The proofs given in the current texts are not conclusive.

173. EXERCISES.

1. Verify the above theorem by counting the number of edges, faces, and vertices in each of the regular figures given in § 170.

2. The following is a form of proof of this theorem which is often given:

Denote the number of vertices, edges, and faces of a polyhedron by V, E, and F, respectively. To prove $E + 2 = V + F$.

Proof: Taking the single face $ABCD$, the number of edges equals the number of vertices, or $E = V$. If another face be annexed, three new edges and two new vertices are added. Hence the number of edges gains one on the number of vertices, as $E = V + 1$. If still another face be added, two new edges and one new vertex are added. Hence $E = V + 2$.

With each new face that is annexed the number of edges gains one on the number of vertices, till but one face is lacking.

The last face increases neither the number of edges nor vertices. Hence, etc.

Show by putting together the faces of a cube in a certain order that the statement in italics need not be true, and hence that the proof is not conclusive.

174. THEOREM. *The sum of the face angles of any convex polyhedron is equal to four times as many right angles, less eight, as the polyhedron has vertices.*

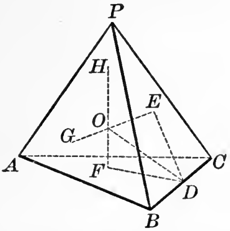
The proof depends on the preceding theorem and is not given here.

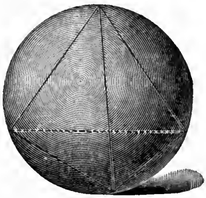
175. EXERCISE.

Verify this theorem by an examination of the regular polyhedrons.

INSCRIPTION OF REGULAR POLYHEDRONS.

176. PROBLEM. *To find a point equally distant from the four vertices of any tetrahedron.*





Given the tetrahedron *P–ABC*.

To find a point O equidistant from P, A, B, C.

Construction. At D, the middle point of BC, construct a plane perpendicular to BC.

The plane will contain the point E, the center of the circle circumscribed about $\triangle PBC$ and also the similar point F in $\triangle ABC$. (Why?)

In the plane DEF draw $EG \perp ED$ and $FH \perp FD$.

Then EG and FH cannot be parallel (why?), and hence meet in some point O.

Also $EG \perp$ to plane PBC and $FH \perp$ plane ABC. (Why?)

Proof: Now show that O is equidistant from P, A, B, C.

177. Definitions. The locus of all points in space equidistant from a given fixed point is a surface called a **sphere.** The fixed point is the **center** of the sphere, and a line-segment from the center to the surface is called a **radius.**

A polyhedron is **inscribed in a sphere** if all its vertices lie in the sphere.

The sphere is also said to be **circumscribed about the polyhedron.**

178. EXERCISES.

1. In the construction of § 176 show that O is the *only* point equidistant from P, A, B, and C.

2. Show that the planes perpendicular to each of the six edges of the tetrahedron at their middle points meet in the point O.

3. Does the construction of § 176 depend upon the tetrahedron being *regular?* Can a sphere be circumscribed about *any* tetrahedron?

4. Is there any limitation on the relative position of four points in order that a sphere may be passed through them?

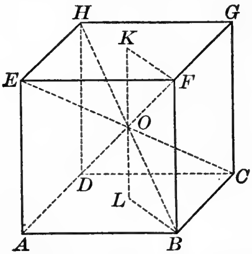
179. Four points not all lying in the same plane are said to **determine a sphere.**

Any other point, taken at random, will not, in general, lie on a sphere determined by four given points.

Hence, while any tetrahedron may be inscribed in a sphere, a polyhedron, in general, cannot.

However, any *regular* polyhedron may be inscribed in a sphere.

180. PROBLEM. *To inscribe a cube in a sphere.*

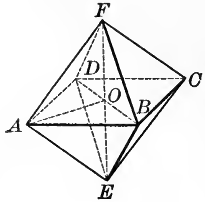


Suggestion. Show that all the diagonals meet in a common point O which is equally distant from all the vertices.

Or show that a perpendicular to one face at its center L meets the opposite face at its center K and is perpendicular to this face also, and that the middle point O of KL is the point required.

181. PROBLEM. *To inscribe an octahedron in a sphere.*

Suggestion. Making use of the construction, § 170 (3), show that O is the point equidistant from A, B, C, D, E, F. Give all the steps in full.



NOTE. The dodecahedron and icosahedron may each be inscribed in a sphere, but the proof in these cases is much more complicated.

182. **EXERCISES.**

1. If in the figure of § 176 the tetrahedron is *regular*, show that $OE = OF$.

2. If in Ex. 1 a sphere is described with center O and radius OE, would it touch the face PBC at any other point than E? (Why?) Would any part of the surface lie on the opposite side of ABC from O? (Why?) Is the same true of each of the other faces?

Such a sphere is said to be **inscribed in the tetrahedron** and the faces are said to be tangent to the sphere. See § 214.

3. In the figure of § 180 show that the point O is equidistant from the six faces of the cube and hence that a sphere may be inscribed.

4. In the figure of § 181 show that the point O is equidistant from the eight faces and hence that a sphere may be inscribed.

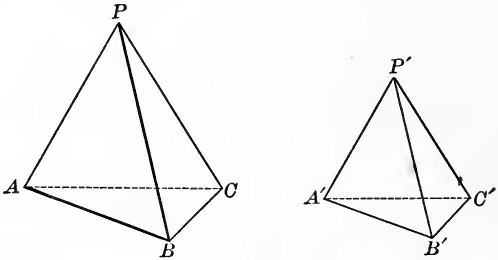
The preceding exercises show that a sphere may be inscribed in three of the regular polyhedrons, and the center in each case is the same as that of the circumscribed sphere. This is true also of the other two regular polyhedrons, but the proof is not so simple as in these cases. In the case of the tetrahedron a sphere may be inscribed *whether it is regular or not;* but if it is not regular, the center is not the same as that of the circumscribed sphere.

SIMILAR POLYHEDRONS.

183. Definitions. Two polyhedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyhedral angles congruent.

Any two parts which are similarly placed are called **corresponding parts**, as corresponding faces, edges, vertices.

184. THEOREM. *Two tetrahedrons are similar if three faces of one are similar respectively to three faces of the other, and are similarly placed.*



Given the tetrahedrons $P–ABC$ and $P'–A'B'C'$ having $\triangle APB \sim \triangle A'P'B'$, $\triangle APC \sim \triangle A'P'C'$, and $\triangle BPC \sim \triangle B'P'C'$.

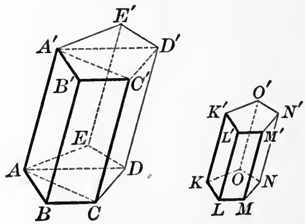
To prove $P–ABC \sim P'–A'B'C'$.

Proof: (1) Show that $\triangle ABC \sim \triangle A'B'C'$.

(2) Show that trihedral $\angle$s P and P' are congruent.

Likewise $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$.

Hence, by definition the polyhedrons are similar.



185. EXERCISES.

1. If the two prisms in the figure are similar, name the pairs of corresponding parts. Likewise for two similar pyramids.

2. Show that a plane parallel to the base of a pyramid cuts off a pyramid similar to the given pyramid.

SUGGESTION. Use the principles of similar triangles and § 66 to show that all the requirements of the definition (§ 183) are fulfilled.

3. Does a plane parallel to the base of the prism cut off a prism similar to the given prism? Prove.

4. Show that two tetrahedrons are similar if they have a dihedral angle in one equal to a dihedral angle in the other and the including faces similar each to each and similarly placed.

5. Show that the total areas of two similar tetrahedrons are in the same ratio as the squares of any two corresponding edges.

6. Show that if each of two polyhedrons is similar to a third they are similar to each other.

186. THEOREM. *The volumes of two similar tetrahedrons are in the same ratio as the cubes of their corresponding edges.*

Given $P\text{-}ABC \sim P'\text{-}A'B'C'$, with volumes V and V'.

To prove that $\dfrac{V}{V'} = \dfrac{\overline{PA}^3}{\overline{P'A'}^3}$.

Proof: We have $\dfrac{V}{V'} = \dfrac{PA \cdot PB \cdot PC}{P'A' \cdot P'B' \cdot P'C'}$. (§ 146)

Now use the properties of similar triangles to complete the proof. Use the figure of § 184.

187. EXERCISES.

1. Two similar tetrahedral mounds have a pair of corresponding dimensions 3 ft. and 4 ft. If one mound contains 40 cu. ft. of earth, how much does the other contain?

2. The edges of a tetrahedron are 3, 4, 5, 6, 7, and 10. Find the edges of a similar tetrahedron containing 64 times the volume.

3. Find what fraction of the altitude of a tetrahedron must be cut off by a plane parallel to the base, measuring from the vertex, in order that the new pyramid thus detached may have one third of the original volume.

188. **Definitions.** Two figures are said to have a **center of similitude** O, if for any two points A and B of the one the lines AO and BO meet the other in two points, A' and B', called **corresponding points**, such that

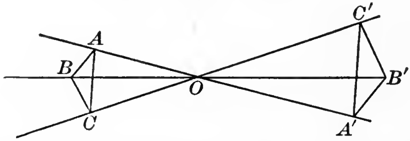
$$\frac{AO}{A'O} = \frac{BO}{B'O}.$$

See figures under §§ 189–194.

189. THEOREM. *Any two figures which have a center of similitude are similar.*

Proof: (1) *Two triangles.*

Given $$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}.$$

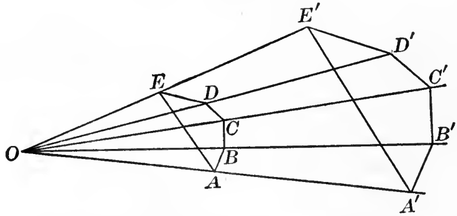


Let the student prove that $\triangle ABC \sim \triangle A'B'C'$.

In case the triangles do not lie in the same plane, use § 34 to show that the corresponding $\angle$s are equal.

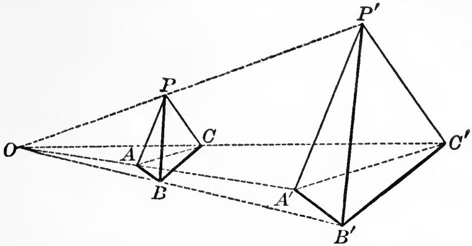
(2) *Two polygons.*

Given $$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}, \text{ etc.}$$



Give the proof both for polygons in the same plane and not in the same plane.

(3) *Two tetrahedrons.*



With the same hypothesis as before, we must prove $\triangle PAB \sim P'A'B'$, $\triangle PBC \sim \triangle P'B'C'$, etc., and then use § 184.

(4) *Any two polyhedrons.*

(*a*) Prove corresponding polygonal faces similar to each other.

(*b*) Prove corresponding polyhedral angles equal to each other.

The last step requires not only equal *face angles* about the vertex, as in the case of the tetrahedron, but also equal *dihedral* angles. Note that two dihedral angles are equal if their faces are parallel right face to right face and left face to left face. (Why?)

(5) Consider *any two figures whatsoever* having a center of similitude.

(*a*) Take any three points A, B, C in one figure and the three corresponding points A', B', C' in the other.

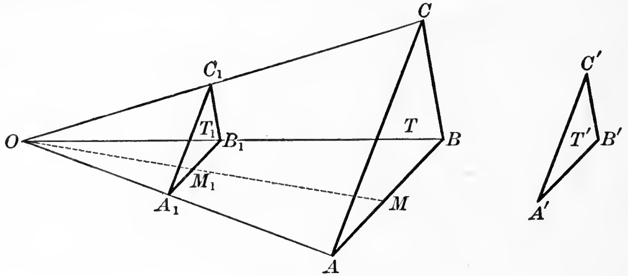
Then AB and $A'B'$, AC and $A'C'$, etc., are called **corresponding linear dimensions**, and the triangles ABC and $A'B'C'$ are **corresponding triangles.**

(*b*) It is clear that any two corresponding linear dimensions have the same ratio as any other two, and that any two corresponding triangles are similar.

In this sense the two figures are said to be **similar.**

190. Definition. The **ratio of similitude** of two similar figures is the common ratio of their corresponding linear dimensions. This ratio is the same as the **distance ratio** of corresponding points from the center of similitude.

191. Theorem. *Two similar triangles may be so placed as to have a center of similitude.*



Given the similar triangles T and T', in which

$$\frac{A'B'}{AB}=\frac{A'C'}{AC}=\frac{B'C'}{BC}.$$

To prove that they may be placed so as to have a center of similitude.

Proof: From any point O draw OA, OB, OC.

On these rays take A_1, B_1, C_1 so that

$$\frac{OA_1}{OA}=\frac{OB_1}{OB}=\frac{OC_1}{OC}=\frac{A'B'}{AB}.$$

Now show the following:

(1) $\triangle T_1 \sim \triangle T$, and hence $\triangle T_1 \sim \triangle T'$.

(2) $\triangle T_1 \cong \triangle T'$.

For this show that $A_1B_1 = A'B'$ by means of the equations $\dfrac{A_1B_1}{AB}=\dfrac{OA_1}{OA}$ and $\dfrac{A'B'}{AB}=\dfrac{OA_1}{OA}$.

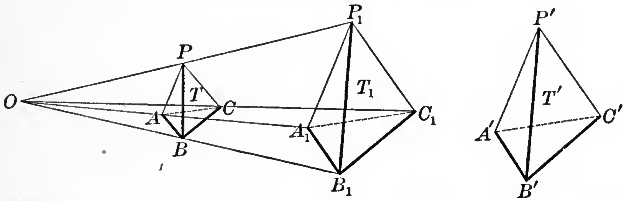
Likewise $A_1C_1 = A'C'$ and $B_1C_1 = B'C'$.

(3) Finally, $$\frac{OM_1}{OM} = \frac{OA_1}{OA},$$

where M and M_1 are any two corresponding points whatever.

Hence O is the required center of similitude.

192. THEOREM. *Two similar tetrahedrons may be so placed as to have a center of similitude.*



Given the similar tetrahedrons T and T'.

To prove that they can be placed so as to have a center of similitude.

Proof: With O as a center of similitude, construct T_1, making $$\frac{OA}{OA_1} = \frac{OB}{OB_1} = \text{etc.} = \frac{AB}{A'B'}.$$

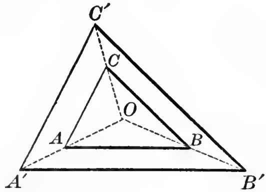
Now show as in § 191 that $T_1 \cong T'$, and hence that T' can be placed in the position T_1 so as to have with T the center of similitude O.

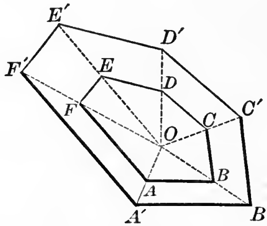
Give all the steps in detail.

193. THEOREM. *Any two similar polyhedrons may be placed so as to have a center of similitude.*

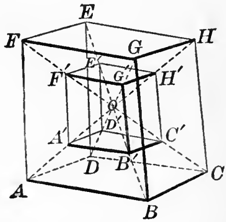
Suggestion. The argument is precisely similar to that of § 192. Give it in full.

194. In the figures for the preceding theorems the center of similitude has been taken *between* the two figures or *beyond* them both. The center may be taken equally well *within* them, as in the following illustrations :





In the case of similar convex polyhedrons with the center of similitude thus placed, the faces are the bases of pyramids whose vertices are all at the center of similitude.



If, further, the polygonal faces be divided into triangles by drawing their diagonals, these triangles become the bases of tetrahedrons, all of whose vertices are at the center of similitude.

Moreover, each inner tetrahedron is similar to its corresponding outer tetrahedron. (Why ?)

The volumes of the two similar polyhedrons are thus composed of the sums of sets of similar tetrahedrons.

195. THEOREM. *The volumes of any two similar polyhedrons have the same ratio as the cubes of their corresponding edges.*

Proof : Place the polyhedrons whose volumes are V and V' so as to have their centers of similitude within them as in the figures of § 194.

Call the volumes of the similar tetrahedrons $T_1, T_2, T_3, \cdots$, and $T_1', T_2', T_3', \cdots$, and let AB and $A'B'$ be two corresponding edges.

Then we have

$$\frac{\overline{AB}^3}{\overline{A'B'}^3} = \frac{T_1}{T_1'} = \frac{T_2}{T_2'} = \frac{T_3}{T_3'} = \cdots. \quad \text{(Why?)}$$

And $$\frac{T_1 + T_2 + T_3 + \cdots}{T_1' + T_2' + T_3' + \cdots} = \frac{T_1}{T_1'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}. \quad \text{(Why?)}$$

But $$T_1 + T_2 + T_3 \cdots = V \text{ and } T_1' + T_2' + T_3' \cdots = V'.$$

Hence, $$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

196. COROLLARY. *The volumes of any two similar solids are in the same ratio as the cubes of any two corresponding linear dimensions.*

This proposition may be rendered evident by noticing that any two similar three-dimensional figures may be built up to any degree of approximation by means of pairs of similar tetrahedrons similarly placed. The proposition then holds of any two corresponding figures used in the approximations.

Note that the ratio of similitude of two similar figures may be obtained from the ratio of any pair of corresponding linear dimensions.

197. EXERCISES.

1. If two coal bins are of the same shape and one is twice as long as the other, what is the ratio of their cubical contents?

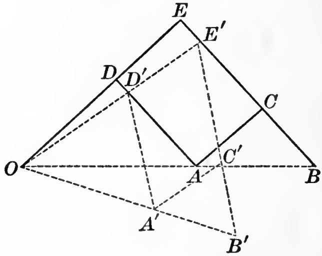
2. What is the ratio of the lengths of the two bins in the preceding example if one holds twice as much coal as the other?

3. Two water tanks are of the same shape. Find the ratio of their capacities if their ratio of similitude is $\frac{2}{5}$.

4. In the preceding what must be the ratio of similitude in order that the ratio of their capacities shall be $\frac{3}{7}$?

APPLICATIONS OF SIMILARITY.

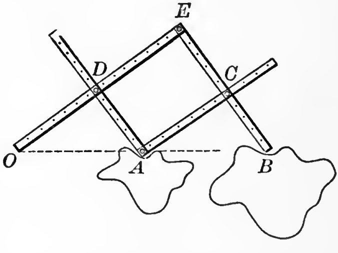
198. The theorem that any two figures which have a center of similitude are similar is the geometric basis of many mechanical contrivances for enlarging or reducing both plane and solid figures; that is, for constructing figures similar to given figures and having a given ratio of similitude with them.



The essential property of all such contrivances is that one point O is kept fixed, while two points A and B are allowed to move so that O, A, and B always remain in a straight line, and so that the ratio $\frac{OA}{OB}$ remains the same. See §§ 432–435 of Plane Geometry.

In the first figure O is a fixed point. Segments OD, CB, and the sides of the parallelogram $ACED$ are of fixed length.

Prove that if B is once so taken on the line EC as to be in the line OA, the points O, A, and B will always remain collinear, and that $\frac{OA}{OB}$ remains a fixed ratio.



In the second figure is shown an ordinary **pantograph** used for copying and at the same time for reducing or enlarging maps, designs, etc. The lengths of the various segments are adjustable, as shown, thus obtaining any desired scale.

The same contrivance may be used for copying figures in *space* and at the same time reducing or enlarging them.

199. Now consider any two similar figures whatever so placed as to have a center of similitude O. We have seen that if points AB and $A'B'$ are corresponding points of the two figures, then the ratio of the **corresponding linear dimensions** AB and $A'B'$ is equal to the ratio of similitude $\frac{m}{n}$ of the two figures.

Also if A, B, C, D and A', B', C', D' are corresponding points, then $\triangle ABC$ and $A'B'C'$, and the tetrahedrons $ABCD$ and $A'B'C'D'$ are similar, and we have

$$\frac{\text{Area } ABC}{\text{Area } A'B'C'} = \frac{m^2}{n^2} \text{ and } \frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{m^3}{n^3}.$$

The points A, B, C and $A'B'C'$ determine two planes, each of which intercepts a certain plane figure in the solid figure to which the points belong. These two plane figures we call **corresponding cross sections.**

We assume without full argument that:—

THEOREM. (1) *The ratio of the areas of any pair of corresponding cross sections or any pair of corresponding surfaces of similar figures is equal to the square of their ratio of similitude,* and

(2) *The ratio of the volumes of any two similar figures is equal to the cube of their ratio of similitude.*

Thus the ratio of the radii or of the diameters of two spheres is their ratio of similitude; likewise the ratio of the lengths or of the diameters of two shells used in gunnery, or the ratio of the heights of two men of similar build.

The fact that the ratio of the areas of corresponding surfaces of similar solids is equal to the *square* of their ratio of similitude, while the ratio of their volumes equals the *cube* of this ratio is one of the most important and far-reaching conclusions of geometry.

SUMMARY OF CHAPTER IV.

1. Describe the five regular solids according to the form and number of their faces. Why can there not be more than these five?

2. Compare the relation of the regular solids to the sphere with that of regular polygons in the plane to the circle.

3. Review the process of construction of the tetrahedron, cube, and octahedron.

4. Form all five regular polyhedrons by cardboard models as indicated in § 170.

5. Make a list of the definitions concerning similar polyhedrons.

6. Make a list of the theorems concerning similar polyhedrons.

7. Explain the relation of two figures which have a center of similitude.

8. What theorem of this chapter is referred to as of unusual importance in the problems and applications?

9. State the applications of this chapter which appeal to you as especially interesting or useful. Return to this question, after studying those which follow.

PROBLEMS AND APPLICATIONS.

1. If it is known that a steel wire of radius r will carry a certain weight w, how great a weight will a wire of the same material carry if its radius is $2r$?

Suggestion. The tensile strengths of wires are in the same ratio as their cross-section areas.

2. Find the ratio of the diameters of two wires of the same material if one is capable of carrying twice the load of the other; three times the load.

3. In a laboratory experiment a heavy iron ball is suspended by a steel wire. In suspending another ball of twice the diameter a wire of twice the radius of the first one is used. Is this perfectly safe if it is known that the first wire will just safely carry the ball suspended from it? Discuss fully.

4. In two schoolrooms of the same shape (similar figures) but of different size, the same *proportion* of the floor space is occupied by desks. Which contains the larger amount of air for each pupil?

5. It is decided to erect a school building exactly like another already built, except that every linear dimension is to be increased by ten per cent; that is, each room is to be ten per cent longer, wider, and higher, and so for all parts of the building. If the air in the ventilating flues flows with the same velocity in the two buildings, in which will the air in a room be entirely renewed the more quickly?

SUGGESTION. Note that the ratio of the amount of air discharged by two flues under such conditions is equal to the ratio of their cross-section areas.

6. If the shells used in guns are similar in shape, find the ratio of the total surface areas of an eight inch and a twelve inch shell.

7. Find the ratio of the weights of the shells in the preceding problem, weights being in the same ratio as the volumes.

8. If a man 5 ft. 9 in. tall weighs 165 lb., what should be the weight of a man 6 ft. 1 in. tall, supposing them to be similar in shape?

9. What is the diameter of a gun which fires a shell weighing twice as much as a shell fired from an eight-inch gun, supposing the shells to be similar bodies?

10. The ocean liner *Mauretania* is 790 feet in length. What must be the length of a ship having twice her tonnage, supposing the boats to be similar in shape?

11. The steamship *Lusitania* is 790 feet long with a tonnage of 32,500, and the *Olympic* is 882 feet long with a tonnage of 45,000. Are these vessels similar in shape? If not, which has the greater capacity in proportion to its length?

12. Supposing two trees to be similar in shape, what is the diameter of a tree whose volume is three times that of one whose diameter is 2 feet? What is the diameter if the volume is 5 times that of the given tree? What if it is n times that of the given tree?

13. Two balloons of similar shape are so related that the total surface area of one is 5 times that of the other. Find the ratio of their volumes.

See page 166 for further applications.

CHAPTER V.

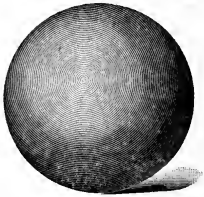
THE SPHERE.

PLANE SECTIONS OF THE SPHERE.

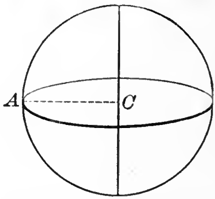
200. Definitions. A **sphere** consists of all points in space which are equally distant from a fixed point, and of these points only. The fixed point is called the center of the sphere. (See § 177.)

A sphere divides space into two parts such that any point which does not lie on the sphere lies within it or outside it.

The sphere may be developed by revolving a circle about a diameter as a fixed axis.



A line-segment joining any two points on a sphere and passing through its center is a **diameter**. A segment joining the center to any point on the sphere is a **radius**.



If the distance from a point to the center of the sphere is less than the radius, the point is **within** the sphere, and if greater than the radius it is **outside** the sphere.

A sphere may be designated by a single letter at its center, or more explicitly by naming its center and radius.

Thus the sphere C means the sphere whose center is C, and the sphere CA is the sphere whose center is C and whose radius is CA.

Two spheres are said to be **equal** if they have equal radii.

201. THEOREM. *All radii of the same sphere or of equal spheres are equal. All diameters of the same sphere or of equal spheres are equal.*

These statements follow directly from the definitions.

202. EXERCISES.

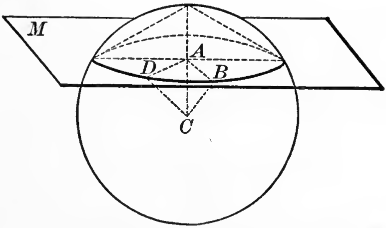
1. How does the definition of a sphere differ from that of a circle? State each in terms of a *locus*.

2. If two spheres have the same center, show that they are either equal or one lies entirely inside the other.

3. In how many points can a straight line meet a sphere?

4. Does every *line* through an interior point of a sphere meet it? In how many points?

203. THEOREM. *A section of a sphere made by a plane is a circle.*



Given a sphere with center C cut by the plane M.

To prove that the points common to the sphere and the plane form a circle.

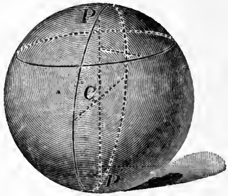
Proof: From the center C draw CA perpendicular to the plane M.

Let B and D be any two points common to the plane and the sphere. Complete the figure, and prove $AB = AD$.

Hence, any two points common to the plane and the sphere are equidistant from A.

How must this proof be modified in case the plane M passes through the center of the sphere?

204. Definitions. A circle is said to be **on a sphere** if all its points lie on the sphere.

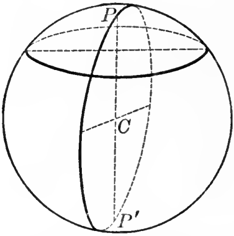


The line perpendicular to the plane of a circle at its center is called the **axis** of the circle.

The points in which the axis of a circle on a sphere meets the sphere are called the **poles of the circle.**

If the plane of a circle on a sphere passes through the centre of the sphere, it is called a **great circle** of the sphere, and if not, it is called a **small circle.**

205. THEOREM. (1) *The axis of any circle on a sphere passes through the center of the sphere.*



(2) *The center of a great circle is the center of the sphere.*

(3) *All great circles are equal and bisect each other.*

(4) *Three points on a sphere determine a circle on the sphere.*

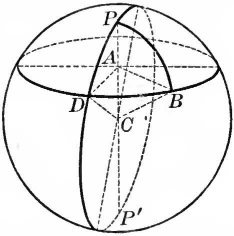
(5) *Through two given points on a sphere there is one and only one great circle unless these points are at opposite ends of a diameter.*

(6) *Every great circle bisects a sphere.*

The proofs of these statements follow easily from the definitions. Let the student give the proofs in detail.

206. Definition. The **distance** between two points on a sphere is the distance measured between these points along the minor arc of the great circle through them.

207. THEOREM. *All points of a circle on a sphere are equidistant from either pole of the circle.*



Given P a pole of the circle whose center is A, and let B and D be any two points on this circle.

To prove that the great circle arcs PB and PD are equal.

Suggestion. Let C be the center of the sphere.

Prove that $\angle ACB = \angle ACD$.

Hence, chord $PB =$ chord PD and $\widehat{PB} = \widehat{PD}$.

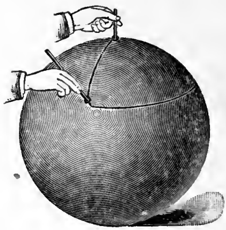
Extend the radius PC to meet the sphere in P'.

Prove that P' is also equidistant from any two points of the given circle.

208. Definition. The common distance from the pole of a circle to all points on it is called the **polar distance of the circle.** One fourth of a great circle is a **quadrant.**

209. COROLLARY 1. *The polar distance of a great circle is a quadrant.*

210. COROLLARY 2. *If a point P is at a quadrant's distance from each of two points not at the extremities of the same diameter, it is the pole of the great circle through these points.*



211. It follows from the preceding theorem that, if a *spherical* blackboard is at hand, circles may be constructed on it by means of crayon and string the same as on a *plane* blackboard. Likewise, curve-legged compasses may be used.

212. EXERCISES.

1. How many small circles can be passed through two points on a sphere? How many great circles? Show why.

2. If two points are at the extremities of the same diameter of a sphere, how many great circles can be passed through these points?

3. What great circles on the earth's surface pass through both poles? If a great circle passes through one pole, must it pass through the other?

4. If P is at a quadrant's distance from each of two points A and B, and if these points are at opposite ends of the same diameter, is P the pole of any circle through A and B? Of how many circles?

5. If A and B are at opposite ends of a diameter, can a small circle be passed through them?

6. If two circles on a sphere have the same poles, prove that their planes are parallel.

7. What is the locus of all points on a sphere at a quadrant's distance from a given point?

8. What is the locus of all points on a sphere at any fixed distance from a given point on the sphere? What is the greatest such distance possible? Discuss fully.

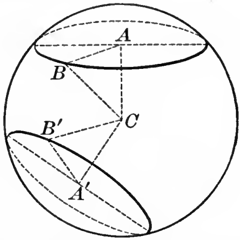
9. If two planes cutting a sphere are parallel, compare the positions of the poles of the circles thus formed.

10. Find the locus of the centers of a set of circles on a sphere formed by a set of parallel planes cutting it.

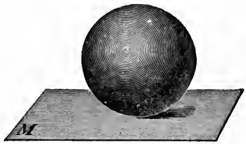
11. AB is a fixed diameter of a sphere. A plane containing AB is made to revolve about it as an axis. Find the locus of the poles of the great circles on the sphere made by this revolving plane. How are the points A and B related to this locus?

213. THEOREM. *If two planes cutting a sphere are equidistant from the center, the circles thus formed are equal; and conversely,*

If two planes cut a sphere in equal circles, the planes are equidistant from the center.



Suggestion. In the figure show that (1) if $CA = CA'$, then $AB = A'B'$, and (2) if $AB = A'B'$, then $CA = CA'$.



214. Definitions. A plane which meets a sphere in only one point is **tangent to the sphere.**

Two spheres are tangent to each other if they have only one point in common.

A line is tangent to a sphere if it contains one and only one point of the sphere.

215. **EXERCISES.**

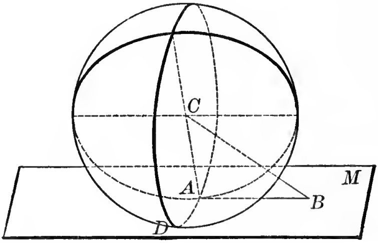
1. If a plane has more than one point in common with a sphere, must it have a circle in common with the sphere?

2. If a plane is tangent to a sphere, how many lines in the plane are tangent to the sphere?

3. Can two spheres be tangent to each other and still one be inside the other?

216. THEOREM. *A plane tangent to a sphere is perpendicular to the radius from the point of tangency; and conversely,*

A plane perpendicular to a radius at its extremity is tangent to the sphere.



Given a sphere C with plane M tangent to it at A.

To prove that CA is perpendicular to the plane M.

Proof: (1) It is only necessary to prove that CA is perpendicular to every line in M through A. (Why?)

Draw any such line AB.

The plane BAC cuts the sphere in a circle. Prove that AB is tangent to this circle, and hence perpendicular to AC.

(2) To prove the converse, note that CA is the shortest distance from C to the plane M. (Why?) And hence that every point of M except A is *exterior* to the sphere.

Hence, M is a tangent plane. (Why?)

217. Definition. A sphere is said to be **inscribed in a polyhedron** if every face of the polyhedron is tangent to the sphere. The polyhedron is also said to be **circumscribed about the sphere.**

218. THEOREM. *A sphere may be inscribed in any tetrahedron.*

The proof is left to the student. See § 182.

219. **EXERCISES.**

1. How many planes may be tangent to a sphere at one point on the sphere? How many lines?

2. Through a given point exterior to a sphere construct a line tangent to the sphere.

SUGGESTION. Let O be the center of the sphere and P the given exterior point. Pass any plane M through P and O. In the plane M construct a line through P tangent to the great circle in which M cuts the sphere.

3. How many lines tangent to a sphere can be constructed from a point outside the sphere?

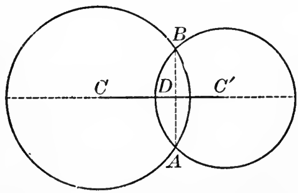
4. Through a given point exterior to a sphere construct a plane tangent to the sphere.

SUGGESTION. As in Ex. 2 construct a line through the given point tangent to the sphere. Through the point of tangency of this line pass a plane tangent to the sphere.

5. How many planes can be passed through a given exterior point tangent to the sphere?

6. How many planes tangent to a sphere can be passed through two given points A and B outside a sphere? Discuss fully if the line AB (1) meets the sphere in two points; (2) is tangent to the sphere; (3) does not meet the sphere.

220. THEOREM. *The intersection of two spheres is a circle.*



Proof: The two intersecting spheres may be developed by rotating about a fixed axis CC' two intersecting circles with centers C and C'.

Let A and B be the two points common to both circles. Then $BA \perp CC'$. (Why?)

As the figure rotates about the line CC', AB remains fixed in length and perpendicular to CC'.

Hence, B traces out a circle. See § 23.

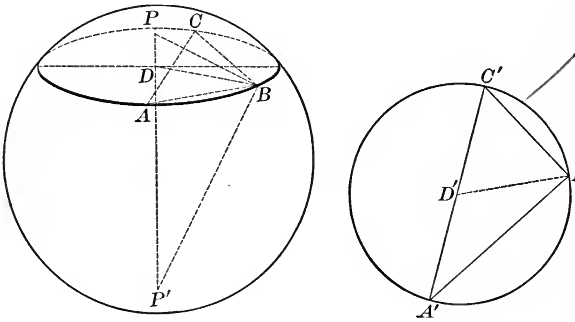
221. PROBLEM. *To find the diameter of a given material sphere.*

With any point P of the sphere as a pole, construct any circle, and on this circle select any three points A, B, C.

Using a pair of compasses, measure the straight line-segments AB, BC, CA, and construct the triangle $A'B'C'$ congruent to ABC.

Let $B'D'$ be the radius of the circle circumscribed about $A'B'C'$.

If PP' is the axis of the circle ABC on the sphere and BD the radius of this circle, then $BD = B'D'$.



This Fig is off A'C' sh not po center

Measure PB by means of the compasses.

Then PBP' is a right triangle, with BD perpendicular to its hypotenuse PP'.

PB and BD being known, we may now *compute* PD from the right triangle PBD and then compute PP' from the similar triangles PBD and $PP'D$, finding $PD : PB = PB : PP'$ or $PD \times PP' = \overline{PB}^2$. See Ex. 3, § 66, Plane Geometry.

The segment PP' may also be found by *geometric* construction; namely, draw a triangle congruent to $P'BP$.

Show how to do this when BP and BD are known.

222. EXERCISES.

1. How many points are necessary to determine a sphere? See § 177.

2. If the center of a sphere is given, how many points on the sphere are required to determine it?

3. If a plane M is tangent to a sphere at a point A, show that the plane of every great circle of the sphere through A is perpendicular to M.

4. Show that the line of centers of two intersecting spheres meets the spheres in the poles of their common circle.

5. Find the locus of the centers of all spheres tangent to a given plane at a given point.

6. Find the locus of the centers of all spheres tangent to a given line at a given point.

7. Find the locus of the centers of all spheres of given radius tangent to a fixed plane.

8. Find the locus of the centers of all spheres of given radius tangent to a fixed line.

9. Find the locus of the centers of all spheres tangent to two given intersecting planes.

10. Find the locus of the centers of all spheres tangent to all faces of a trihedral angle.

11. Show that two spheres are tangent if they meet on their line of centers. Distinguish two cases. Compare § 209, Plane Geometry.

12. State and prove the converse of the preceding proposition.

13. In plane geometry how many circles can be drawn through a given point tangent to a given line at a given point?

14. If a given sphere is tangent to a given plane M at a given point A, how many points on the sphere are required to determine it?

SUGGESTION. Suppose one point P given. Pass a plane through $P \perp$ to plane M at A. Is there only one such plane? Discuss fully.

15. Describe the set of all lines in space whose distances from the center of a sphere are all equal to the radius of the sphere.

16. Describe the set of all planes whose distances from the center of a sphere are all equal to the radius of the sphere.

TRIHEDRAL ANGLES AND SPHERICAL TRIANGLES.

223. **Definitions.** When two curves meet, they are said to **form an angle**; namely, the angle made by the tangents to the curves at their common point.

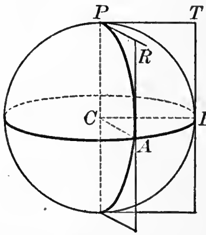
Any two planes through the center of a sphere cut out two great circles which intersect in two points and form four **spherical angles** about each of these points. Two of these angles with a common vertex are either **adjacent** or **vertical** in the same sense as the angles formed by two intersecting straight lines.

A spherical angle is **acute**, **right**, or **obtuse** according to the form of the angle between the tangents to its sides (arcs) at their common point.

Any two circles on a sphere which meet, whether great circles or not, form angles according to the above definition.

Only angles formed by great circles are considered in this book and the expression **spherical angle** will be understood to refer to such angles. Only angles greater than zero and not greater than two right angles are considered.

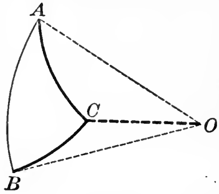
A spherical angle may be denoted by a single letter or by three letters, as in the case of a plane angle.



224. THEOREM. *A spherical angle is measured by an arc of the great circle whose pole is the vertex of the angle and which is intercepted by the sides of the angle.*

Suggestion. Show that $\overset{\frown}{AB}$ measures the dihedral angle formed by the planes PAC and PBC and that $\angle BCA = \angle TPR$.

225. Definitions. The section of a sphere made by a convex trihedral angle, whose vertex is at the center of the sphere, is called a **spherical triangle.**



The face angles of the trihedral angle are measured by the *sides* (arcs) of the spherical triangle, and its dihedral angles are equal to the *angles* of the spherical triangle.

Since each face angle of a trihedral angle is less than two right angles, it follows that each side of a spherical triangle is less than a semicircle.

226. EXERCISES.

1. Show that a spherical angle is equal to the plane angle of the dihedral angle formed by the planes of the great circles whose arcs are the sides of the spherical angle. See figure under § 224.

2. Prove that vertical spherical angles are equal.

3. Prove that the sum of the spherical angles about a point is four right angles.

4. At what angle does a meridian on the earth's surface intersect the equator?

5. Denote by P the North Pole on the earth's surface. Consider any two meridians forming an angle of one degree at P and meeting the equator in points A and B, respectively. What is the sum of the angles of the spherical triangle PAB? Compare with the sum of the angles of a plane triangle.

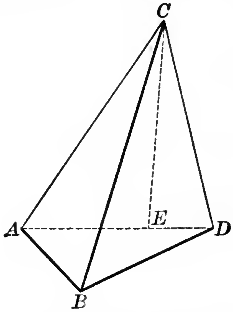
6. Is it possible to construct a spherical triangle each of whose angles is a right angle?

SUGGESTION. Consider two meridians forming a right angle at P. Such a triangle is called a trirectangular triangle.

7. In a trirectangular spherical triangle what is the length of each side in terms of degrees?

The question as to whether or not the sum of the angles of a spherical triangle is ever equal to two right angles is answered in § 257.

227. THEOREM. *The sum of two face angles of a trihedral angle is greater than the third face angle.*



Proof: Connect points A and D on two sides.

Suppose not all three face angles are equal and that $\angle ACD > \angle ACB$.

Construct CE in the face ACD, making $\angle ACE = \angle ACB$.

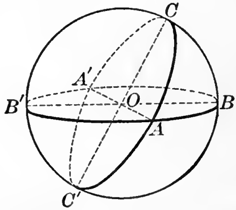
Lay off $CB = CE$ and draw AB and BD.

Now show that (1) $AB = AE$, (2) $AD < AB + BD$, (3) $ED < BD$, (4) $\angle ECD < \angle BCD$.

Hence, $\angle ACD < \angle ACB + \angle BCD$.

228. COROLLARY 1. *The sum of two sides of a spherical triangle is greater than the third side.*

229. COROLLARY 2. *The sum of the three sides of a spherical triangle is less than a great circle.*

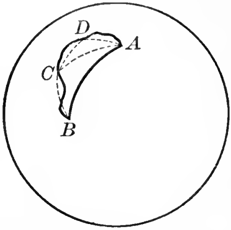


Proof: $\overset{\frown}{ACA'} + \overset{\frown}{ABA'} =$ a great circle.
But $\overset{\frown}{AC} < \overset{\frown}{ACA'}$ and $\overset{\frown}{AB} < \overset{\frown}{ABA'}$,
and $\overset{\frown}{BC} < \overset{\frown}{BA'} + \overset{\frown}{CA'}$ by Corollary 1.
Hence, $\overset{\frown}{AC} + \overset{\frown}{CB} + \overset{\frown}{BA} <$ a great circle.

230. COROLLARY 3. *State and prove the theorems on trihedral angles corresponding to Corollaries* 1 *and* 2.

231. THEOREM. *The shortest distance on a sphere between two of its points is measured along the minor arc of a great circle passing through these points.*

Proof: Let A and B be any two points on a sphere, AB the minor arc of a great circle through them, and $ADCB$ any other curve on the sphere connecting A and B with points C and D on it in the order $ADCB$. Neither C nor D is on the arc AB.



Draw the great circle arcs AD, AC, DC, and CB. Then by § 228 $\overset{\frown}{AC} + \overset{\frown}{CB} > \overset{\frown}{AB}$ and $\overset{\frown}{AD} + \overset{\frown}{DC} > \overset{\frown}{AC}$.

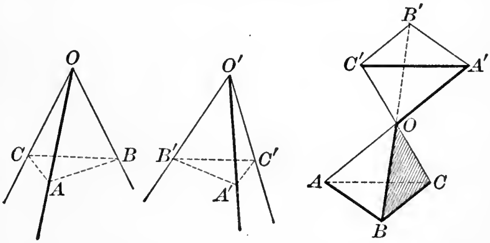
Hence, $\overset{\frown}{AD} + \overset{\frown}{DC} + \overset{\frown}{CB} > \overset{\frown}{AB}$.

Continuing in this manner, we obtain a succession of paths, each longer than the preceding. But by this process we get closer and closer to the length of the curve ACB.

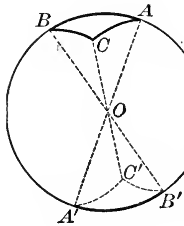
Hence, it must be greater than that of $\overset{\frown}{AB}$.

232. Definitions. Two trihedral angles are **symmetrical** one to the other if the face angles and the dihedral angles of one are equal respectively to the face angles and the dihedral angles of the other, but arranged in the opposite order.

Similarly, two spherical triangles are symmetrical one to the other if the sides and the angles of one are equal respectively to the sides and the angles of the other, but arranged in the opposite order.



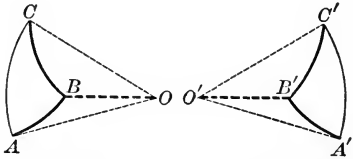
233. THEOREM. *If the radii drawn from the vertices of a spherical triangle are extended, they meet the sphere in the vertices of a triangle symmetrical to the given triangle.*



The proof is left to the student.

234. COROLLARY. *State and prove the corresponding theorem on trihedral angles.*

235. THEOREM. *Two trihedral angles having their vertices at the center of the same or of equal spheres intercept congruent spherical triangles if the trihedral angles are congruent, and symmetrical spherical triangles if they are symmetrical.*



This is an immediate consequence of §§ 225, 232.

236. COROLLARY. *If in two spherical triangles three sides of one are equal respectively to three sides of the other, and arranged in the same order, the triangles are congruent.* See § 66.

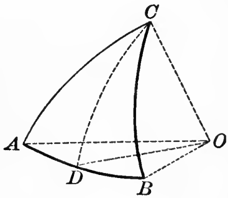
237. THEOREM. *If the face angles of one trihedral angle are equal respectively to the face angles of another, but arranged in the opposite order, the trihedral angles are symmetrical.*

Proof: By § 64 the dihedral angles of one are equal respectively to those of the other. Now verify that these parts are arranged in the opposite order.

238. COROLLARY 1. *If in two spherical triangles the three sides of one are equal to three sides of the other, but arranged in the opposite order, the triangles are symmetrical.*

239. **Definition.** A spherical triangle is isosceles if two sides are equal.

240. COROLLARY 2. *The angles opposite the equal sides of an isosceles spherical triangle are equal.*



Suggestion. Let AC and BC be the equal sides. Draw $\overset{\frown}{CD}$ to the middle point of $\overset{\frown}{AB}$.

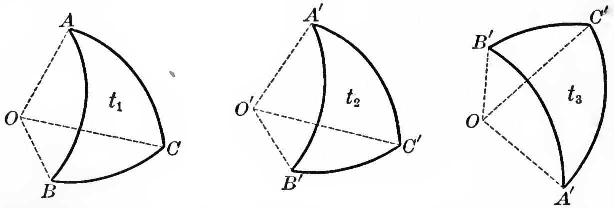
241. COROLLARY 3. *If two isosceles spherical triangles are symmetrical, they are congruent, and conversely.*

242. THEOREM. *If two trihedral angles are symmetrical to the same trihedral angle, they are congruent.*

Suggestion. Show that the corresponding parts must be arranged in the same order.

243. COROLLARY. *State and prove the corresponding theorem for spherical triangles.*

244. THEOREM. *Two spherical triangles having two sides and the included angle of one equal respectively to two sides and the included angle of the other are congruent, if the given parts are arranged in the same order, and symmetrical, if they are arranged in the opposite order.*



Proof: If the given parts are arranged in the same order, the proof may be made by superposition exactly as in § 32, Plane Geometry.

If the given parts are arranged in the opposite order, proceed as follows:

Denote the given triangles by t_1 and t_2. Construct a spherical triangle t_3 symmetrical to t_1. Then by § 243 t_2 and t_3 are congruent. Hence, if t_1 is symmetrical to t_3, it must be symmetrical to t_2.

245. COROLLARY. *State and prove the corresponding theorem for trihedral angles.*

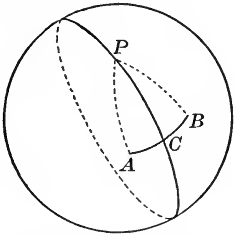
246. EXERCISES.

1. Compare fully the theorems on the congruence of plane triangles and of trihedral angles. Is there any theorem in either case for which there is no corresponding theorem in the other?

2. Compare in the same manner the theorems on the congruence of plane triangles and of spherical triangles.

3. Compare in the same manner the theorems on the congruence of trihedral angles and of spherical triangles.

247. THEOREM. *The locus of all points on a sphere equidistant from two fixed points on the sphere is a great circle bisecting at right angles the great circle arc connecting the two given points.*



Proof: Let C be the middle point of $\widehat{AB}$.

(*a*) If $\widehat{AP} = \widehat{BP}$ prove that $\triangle$ ACP and BCP are symmetrical, and hence, $\angle ACP = \angle BCP =$ rt. $\angle$.

(*b*) If $\angle ACP = \angle BCP$, prove that $\widehat{AP} = \widehat{BP}$.

Why are steps (*a*) and (*b*) both needed?

248. EXERCISES.

1. If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

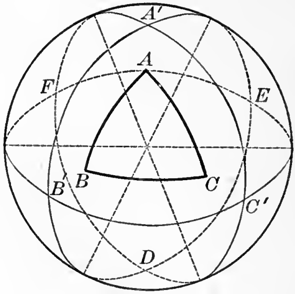
2. If two face angles of a trihedral angle are equal, it is congruent to its symmetrical trihedral angle.

3. Show how to find a pole of the circle through three given points on the sphere.

SUGGESTION. Let the given points be A, B, C. By § 207 the pole of the circle is equidistant from A, B, and C. Connect A and B by an arc of a great circle and construct another arc of a great circle bisecting $\widehat{AB}$ perpendicularly. Similarly construct a perpendicular bisector of $\widehat{BC}$. The points in which these two arcs meet will be the poles of the circle through A, B, and C.

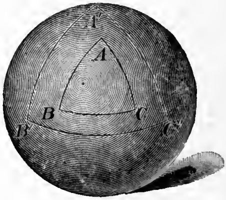
State this argument in full detail.

POLAR TRIANGLES.



249. Definition. With the vertices A, B, C of a spherical triangle as poles, construct three great circles. Each of these circles meets each of the others in two points, thus forming eight spherical triangles, as shown in the figure, namely, $A'B'C'$, $A'B'F$, $B'C'D$, $C'A'E$, $A'EF$, $B'DF$, $C'DE$, and DEF.

There is one and only one of these, namely, $A'B'C'$, such that A and A' are on the same side of $\widehat{B'C'}$, B and B' on the same side of $\widehat{A'C'}$, and C and C' on the same side of $\widehat{A'B'}$.



The triangle $A'B'C'$ as thus described is the **polar triangle** of ABC.

250. EXERCISES.

1. In the figure the parts of the great circles which are supposed to be on the front side of the figure are given in solid lines while the parts on the back side are dotted. Study the figure with care and state which triangles are entirely on the front side, which are entirely on the back side of the sphere, and which are partly on the front side and partly on the rear side of the sphere.

2. Show that the points A' and D cannot be on the same side of the circle through $B'C'$.

SUGGESTION. Can the two extremities of a diameter lie in the same hemisphere?

3. If A is a pole of the great circle through $B'C'$ and if A' is on the same side of this circle as A, show that A and A' are less than one quadrant's distance apart.

251. THEOREM. *If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.*

Proof: It is required to prove (1) that A' is the pole of $\overset{\frown}{BC}$, B' the pole of $\overset{\frown}{AC}$, and C' the pole of $\overset{\frown}{AB}$, and also (2) that A and A' lie on the same side of $\overset{\frown}{BC}$, B and B' on the same side of $\overset{\frown}{AC}$, and C and C' on the same side of $\overset{\frown}{AB}$.

(1) To prove that A' is the pole of $\overset{\frown}{BC}$ we need only to show that A' is at a quadrant's distance from two points in $\overset{\frown}{BC}$. Why?

Now A' is at a quadrant's distance from B because B is the pole of $\overset{\frown}{A'C'}$. A' is also at a quadrant's distance from C because C is the pole of $\overset{\frown}{A'B'}$. Hence, A' is the pole of $\overset{\frown}{BC}$.

Similarly, B' is the pole of $\overset{\frown}{AC}$ and C' the pole of $\overset{\frown}{AB}$.

(2) To show that A and A' lie on the same side of the circle BC, we note that since A is the pole of the circle $B'C'$ and A lies on the same side of this circle with A', then A and A' are at less than a quadrant's distance. Hence, it follows that if A' is at a quadrant's distance from BC, A and A' must be on the same side of BC.

In like manner we show that B and B' lie on the same side of AC and C and C' on the same side of AB.

252. Definition. If ABC and $A'B'C'$ are polar triangles, and if A is a pole of $\overset{\frown}{B'C'}$, then $\angle A$ and $\overset{\frown}{B'C'}$ are said to be **corresponding parts.**

253. EXERCISES.

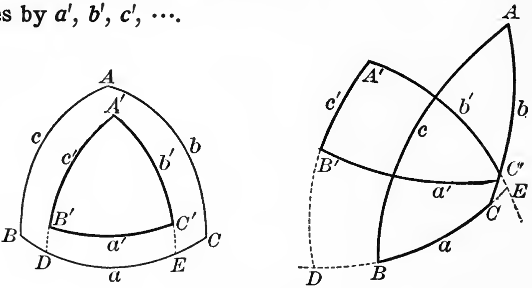
1. In the two polar triangles ABC and $A'B'C'$ name all pairs of corresponding parts.

2. Is there any spherical triangle such that its polar triangle is identical with the given triangle?

3. If one side of a spherical triangle is greater than a quadrant show that it is cut by two circles in the construction of its polar triangle.

254. THEOREM. *The sum of the measures of an angle of a spherical triangle and the corresponding arc of its polar triangle is* 180°.

Given the polar triangles ABC and $A'B'C'$. Denote the measures in degrees of the angles by A, B, C, $\cdots$ and of the corresponding sides by a', b', c', $\cdots$.



To prove that

$$A+a'=180^\circ \qquad A'+a=180^\circ$$
$$B+b'=180^\circ \qquad B'+b=180^\circ$$
$$C+c'=180^\circ \qquad C'+c=180^\circ$$

Proof: Extend (if necessary) arcs $A'B'$ and $A'C'$ till they meet the great circle BC in points D and E, respectively. Then arc DE is the measure of $\angle A'$.

Also $\widehat{BE}=90^\circ$, and $\widehat{DC}=90^\circ$. (Why?)

But $\widehat{BE}+\widehat{DC}=\widehat{BC}+\widehat{ED}=a+A'$.

Hence, $A'+a=180^\circ$.

Complete the proof for the other cases.

255. COROLLARY. *If two spherical triangles are congruent or symmetrical, their polar triangles are congruent or symmetrical.*

256. **EXERCISES.**

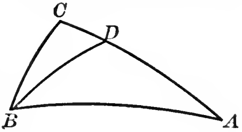
1. Does the above proof apply to the second figure?

2. If two angles of a spherical triangle are equal, it is isosceles.

SUGGESTION. Use § 254, § 240, and again § 254.

3. State and prove the theorem on trihedral angles corresponding to the preceding.

4. If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the greater side being opposite the greater angle.



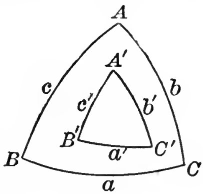
SUGGESTION. In the triangle ABC let $\angle B > \angle A$. Draw $\overset{\frown}{BD}$, making $\angle ABD = \angle A$.

Then, $\qquad \overset{\frown}{AD} = \overset{\frown}{BD}$, and $\overset{\frown}{BD} + \overset{\frown}{DC} > \overset{\frown}{BC}$.

Hence, show that $\qquad \overset{\frown}{AC} > \overset{\frown}{BC}$.

5. State and prove a theorem on trihedral angles corresponding to the preceding.

257. THEOREM. *The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.*



Given the spherical triangle *ABC*.

To prove that (1) $\angle A + \angle B + \angle C < 6$ rt. angles.

(2) $\angle A + \angle B + \angle C > 2$ rt. angles.

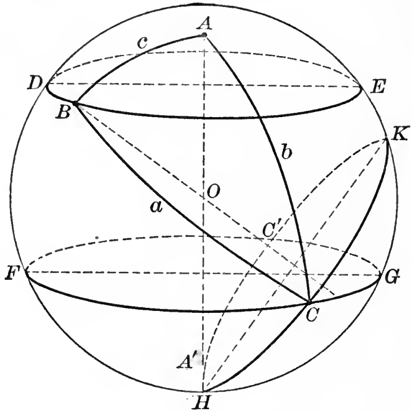
Proof: Construct the polar triangle $A'B'C'$, with sides a', b', c'.

(1) By § 254 $\angle A + \angle B + \angle C + a' + b' + c' = 6$ rt. angles. Since $a' + b' + c'$ is greater than zero, it follows that $\angle A + \angle B + \angle C < 6$ rt. angles.

(2) Using § 229, show that $\angle A + \angle B + \angle C > 2$ rt. $\measuredangle$s.

258. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

259. Problem. *On a given sphere to construct a spherical triangle when its sides are given.*



Solution. Let O be the given sphere, and a, b, c the arcs of the required triangle, and let AA' be any diameter of the sphere. With A as a pole, construct circles DBE and FCG whose polar distances from A are c and b respectively.

With B as a pole, construct a circle HCK, whose polar distance from B is a. Then construct the three great circle arcs, AB, BC, CA. ABC is the required triangle.

260. EXERCISES.

1. What restrictions if any is it necessary to impose upon the three given sides of the triangle in § 259? (§§ 228, 229.)

2. In plane geometry two congruent triangles may be constructed upon the same base and on the same side of it. Is a corresponding construction possible on the sphere?

3. If in the above construction each of two sides of the required triangle is very great, that is, nearly a semicircle, show from the construction that the third side must be very small.

4. If one side of the proposed triangle in § 259 were equal to or greater than 180°, why would that make the construction impossible?

261. Problem. *To construct a spherical triangle when its three angles are given.*

Solution. Let the three given angles be A, B, C, and let a', b', c' be arcs such that $a' + \angle A = 180°$, $b' + \angle B = 180°$, $c' + \angle C = 180°$. Then the triangle whose arcs are a', b', c' will be the polar triangle of the required triangle. This latter triangle $A'B'C'$ may be constructed by the method of § 259. Then construct the polar triangle of $A'B'C'$, which will be the required triangle.

Give reasons in full for each step.

262. Problem. *To construct a trihedral angle when its face angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 259.

Give the construction in full and prove each step.

263. Problem. *To construct a trihedral angle when its dihedral angles are given.*

Solution. Construct the corresponding spherical triangle by the method of § 261.

Give reasons in full for each step.

264. **EXERCISES.**

1. If two spherical triangles having angles respectively equal are constructed on the same sphere, how are these triangles related? Prove.

2. If two trihedral angles with face angles respectively equal are constructed as in § 262, how are they related? Prove.

3. If two trihedral angles each with the same dihedral angles are constructed as in § 263, how are the trihedral angles related? Prove.

4. What restrictions if any must be placed upon the given angles A, B, C in § 261? Compare Ex. 1, § 260.

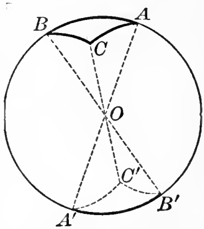
5. What restrictions if any are needed in Exs. 2 and 3?

265. THEOREM. *Two spherical triangles having two angles and the included side of one equal respectively to two angles and the included side of the other are congruent, if the given equal parts are arranged in the same order, and symmetrical, if arranged in the opposite order.*

Proof: By §§ 254 and 244 the polar triangles of the given triangles are congruent or symmetrical. Hence, by § 255, the given triangles themselves are congruent or symmetrical.

266. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

267. THEOREM. *Two spherical triangles having the angles of one equal respectively to the angles of the other are congruent, if the equal angles are arranged in the same order, and symmetrical, if they are arranged in the opposite order.*



Proof: By §§ 254, 236, and 238 the polar triangles of the given triangles are equal or symmetrical. Hence, by § 255, the given triangles themselves are equal or symmetrical.

268. COROLLARY. *State and prove the theorem on trihedral angles which corresponds to the preceding.*

269. Is there a theorem on plane triangles corresponding to that of § 267?

270. EXERCISES.

1. If the sides of a spherical triangle are 60°, 80°, 120°, find the angles of its polar triangle.

2. If the angles of a spherical triangle are 72°, 104°, 88°, find the sides of the polar triangle.

3. If a triangle is isosceles, prove that its polar triangle is isosceles.

4. If each side of a spherical triangle is a quadrant, describe its polar triangle.

5. In case each side of spherical triangle ABC is a quadrant, show that the eight triangles formed by drawing the great circles whose poles are A, B, and C are all congruent.

6. Show that for any triangle the construction of the polar triangle gives four pairs of symmetrical triangles.

7. If a triangle ABC is isosceles, show that of the eight triangles of Ex. 5 there are four pairs of congruent triangles.

8. If the triangle ABC is not isosceles, show that of the pairs of triangles proved congruent in Ex. 7 none are now congruent.

9. If the angles of a spherical triangle are 70°, 80°, and 110° respectively, find the sides of each of the eight triangles formed by the polar construction.

10. Is it possible to construct a spherical triangle whose angles are 50°, 60°, 120°?

11. Is it possible to construct a spherical triangle whose angles are 60°, 120°, 150°?

Suggestion. Consider the polar triangle of such triangle.

12. Consider the questions on trihedral angles corresponding to the two preceding.

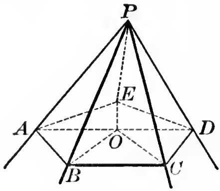
13. If the sides of a spherical triangle are 75°, 95°, and 115° respectively, find the angles of each triangle formed by the polar construction.

14. How can the theorem of § 257 be used to prove that a side of a spherical triangle cannot be as great as a semicircle?

15. If it is given that a spherical triangle is equilateral, can we infer from the theorems thus far proved that its polar triangle is equilateral?

POLYHEDRAL ANGLES AND SPHERICAL POLYGONS.

271. THEOREM. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Proof: Let $ABCDE$ be a polygonal section of the given polyhedral angle. The number of triangles thus formed having P for a vertex is equal to the number of face angles of the polyhedral angle.

Let O be any point in the base, and draw OA, OB, OC, etc.

Then $\angle PBA + \angle PBC > \angle ABC$, and $\angle PCB + \angle PCD > \angle BCD$, and so on. (Why?)

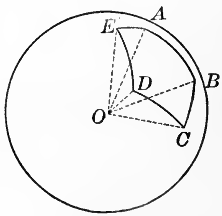
But the sum of the ∠s of the △s OAB, OBC, etc., is equal to the sum of the ∠s of the △s PAB, PBC, etc.

Hence, $\angle APB + \angle BPC + \cdots < \angle AOB + \angle BOC + \cdots$.

But the sum of the ∠s about O is four right angles.

Therefore, the sum of the face angles of the polyhedral angle is less than four right angles.

272. **Definition.** The section of a sphere made by a convex polyhedral angle whose vertex is the center of the sphere is called a **spherical polygon.**



Since a plane may be passed through the vertex of a polyhedral angle such that the polyhedral angle lies entirely on one side of it, it follows that a spherical polygon lies within one hemisphere.

273. Corollary. *State and prove the theorem on spherical polygons which corresponds to that of* § 271.

274. Theorem. *The sum of the angles of a spherical polygon of n sides is greater than $2(n-2)$ right angles and less than $2n$ right angles.*

Proof: Divide the polygon into $n-2$ triangles.

Hence, by § 257 the sum of the angles is greater than $2(n-2)$ right angles.

Since the polygon has n angles, and since each angle is less than two right angles, it follows that their sum is less than $2n$ right angles.

275. Corollary. *State and prove the theorem on polyhedral angles which corresponds to the preceding.*

AREAS OF SPHERICAL POLYGONS.

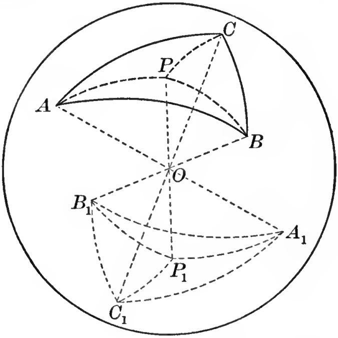
276. Definitions. A spherical polygon divides a sphere into two parts, an **exterior** and an **interior**, so that every path on the sphere passing from one to the other must cross the polygon.

We have seen (§ 272) that any spherical polygon lies within one hemisphere. The interior is that one of the two parts which lies entirely within this hemisphere.

Two spherical polygons are said to **inclose equal areas** or **to be equal** if they are congruent, or if they can be divided into polygonal surfaces which are congruent in pairs.

The terms *spherical triangle*, *spherical polygon*, are sometimes used to refer to the part of the sphere inclosed by these figures. The context will always indicate clearly in which sense they are used.

277. THEOREM. *Two symmetrical spherical triangles are equal.*



Proof: Let ABC and $A'B'C'$ be the given triangles.

Extend the radii OA, OB, OC to meet the sphere in A_1, B_1, C_1, thus forming a triangle symmetrical to $\triangle ABC$ (§ 233), and hence congruent to $\triangle A'B'C'$ (§ 243).

Let P be a pole of the circle through A, B, C. Extend PO to meet the sphere in P_1. Draw $\frown{PA}$, $\frown{PB}$, $\frown{PC}$, and $\frown{P_1A_1}$, $\frown{P_1B_1}$, and $\frown{P_1C_1}$.

Suppose that P lies within $\triangle ABC$.

Now prove that $\triangle PAB \cong \triangle P_1A_1B_1$, $\triangle PAC \cong \triangle P_1A_1C_1$, $\triangle PBC \cong P_1B_1C_1$. Note that these triangles are isosceles.

Hence, show that $\triangle ABC = \triangle A_1B_1C_1$, and therefore $\triangle ABC = \triangle A'B'C'$.

278. **Definitions.** A **lune** is a figure formed by two great semicircles having the same end-points. The angle between these semicircles is the **angle of the lune.**

A **birectangular spherical triangle** is one having two right angles.

If one angle of a birectangular triangle is 1°, the triangle incloses one of 720 equal parts of the sphere. The area

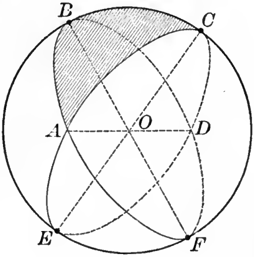
inclosed by such a triangle is called a **spherical degree** and is used as a unit of measure of areas inclosed by spherical polygons.

In a similar manner we define a **spherical minute** and a **spherical second.**

279. THEOREM. *The area inclosed by a lune in terms of spherical degrees is twice the angle of the lune.*

280. Definition. The number of spherical degrees by which the sum of the angles of a spherical triangle exceeds 180° is called the **spherical excess** of the triangle.

281. THEOREM. *The area of a spherical triangle in terms of spherical degrees is equal to its spherical excess.*



Proof : We are to show that area $\triangle ABC = \angle A + \angle B + \angle C - 180°$.

Consider the lunes $ACDB$, $CAEB$, $BCFA$.

We have

$\triangle ABC + \triangle BCD = ACDB = 2 \angle A.$

$\triangle ABC + \triangle BAE = CAEB = 2 \angle C.$

$\triangle ABC + \triangle CFA = BCFA = 2 \angle B.$

Hence, adding, $3 \triangle ABC + \triangle BCD, BAE, CFA = 2(\angle A + \angle B + \angle C)$.

Now $\triangle BCD$ and AEF are symmetrical and have equal areas.

Hence,

$2 \triangle ABC + \triangle ABC, AEF, BAF, CFA = 2(\angle A + \angle B + \angle C).$

But $\triangle ABC, AEF, BAF, CFA$ together constitute a hemisphere or 360 spherical degrees.

Hence, $2 \triangle ABC + 360° = 2(\angle A + \angle B + \angle C).$

Solving, $\triangle ABC = \angle A + \angle B + \angle C - 180°.$

282. Definition. The spherical excess of a spherical polygon is the sum of its angles less $(n-2)180°$, where n is the number of sides of the polygon.

283. THEOREM. *The area of a spherical polygon in terms of spherical degrees is equal to its spherical excess.*

Proof: Join one vertex of the polygon to each non-adjacent vertex, thus forming $n-2$ spherical triangles.

Now prove that the sum of the spherical excesses of these triangles is the spherical excess of the polygon and thus complete the proof.

284. **EXERCISES.**

1. What is the area in spherical degrees of a birectangular triangle one of whose angles is 54°? If one angle is 79° 30′; 106°; 14′; 36″?

2. What is the area inclosed by a lune whose angle is 45°? Note that the lune may be divided into two birectangular triangles. What is the third angle of each?

3. Between what limits is the sum of the angles of a spherical polygon of eight sides?

4. If the sum of the angles of a spherical polygon is 11 right angles, what is known about the number of its sides?

5. If the sum of the angles of a spherical polygon is 14 right angles, what is known about the number of its sides?

6. The sides of a spherical ~~polygon~~ are 85°, 95°, 110°. Find the area of each of the eight triangles formed by the polar construction from this triangle.

7. The area of a spherical triangle is 74 spherical degrees. One angle is 105°. Of the other two angles one is twice the other. Find all the angles of the triangle.

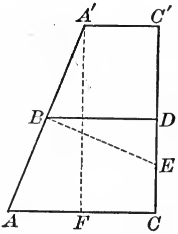
triangle

AREA AND VOLUME OF THE SPHERE.

285. In §§ 276–283 we discussed the areas inclosed by spherical polygons in terms of a unit directly applicable to the sphere, namely, the **spherical degree.**

We now consider the area of the sphere in terms of a plane unit of measure. It is clear that such measurement can be *approximate* only, since no plane segment however small will coincide with the spherical surface. Similarly, if the cube is used as the unit of volume, the measurement of the volume inclosed by a sphere must be *approximate*, since no set of cubes however small can be made exactly to coincide with a sphere.

The two following theorems are needed:



286. THEOREM. *The lateral area of a frustum of a right circular cone is equal to the altitude of the frustum multiplied by the length of a circle whose radius is the perpendicular distance from a point in the axis of the frustum to the middle point of an element.*

Given AA' **an element of the frustum whose middle point is** B^1**,** CC' **the axis of the frustum,** $BD \perp CC'$ **and** $EB \perp AA'$**.**

To prove that the lateral area is equal to $2\pi \cdot EB \cdot CC'$.

Proof: By § 164, the lateral area is $2\pi \cdot BD \cdot AA'$.

Hence, we must show that $EB \cdot CC' = BD \cdot AA'$.

To do this, draw $A'F \perp AC$ and show that $\triangle AFA' \sim \triangle EDB$.

287. COROLLARY. *The lateral area of a cone is equal to its altitude times the length of a circle whose radius is the perpendicular from a point in the axis to the middle point of an element.*

288. THEOREM. *Given a fixed line through the vertex of a triangle, but not crossing it. The volume swept out by the triangle as it rotates about the fixed line as an axis is equal to the numerical measure of the area generated by the side opposite the fixed vertex multiplied by one third the altitude upon that side.*

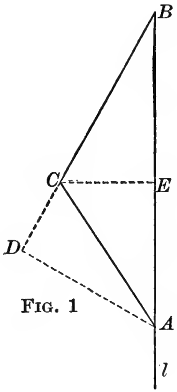


FIG. 1

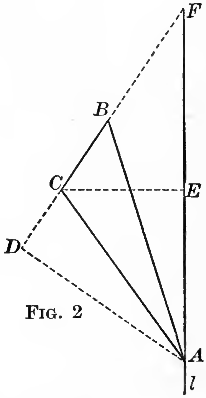


FIG. 2

Given △ *ABC* with vertex *A* in the fixed line *l*.

CASE 1. *When one side as AB lies in line l.* (Fig. 1.)

Draw $AD \perp BC$, or BC produced, and $CE \perp AB$.

Then Vol. $ABC =$ Vol. $AEC +$ Vol. BEC

$$= \frac{\pi}{3}\, \overline{CE}^2 \cdot AB. \quad \text{(Why?)}$$

But $\overline{CE}^2 \cdot AB = CE \cdot CE \cdot AB = CE \cdot BC \cdot AD.$ (Why?)

And by § 158, $\pi \cdot CE \cdot BC$ is the area swept out by BC.

Hence, Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by BC).

CASE 2. *When neither AB nor AC lies in line l.* (Fig. 2.)

Produce CB to meet l in F and draw AD and CE as before.

425

Then Vol. $ABC =$ Vol. $AFC -$ Vol. AFB.

By Case 1 Vol. $AFC = \frac{1}{3} AD \cdot$ (area generated by FC),

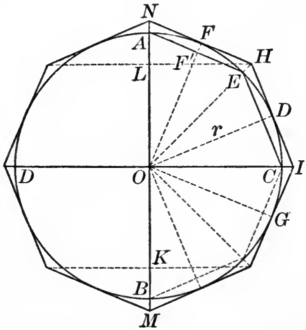
and Vol. $AFB = \frac{1}{3} AD \cdot$ (area generated by FB).

Hence, Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by $FC - FB$),

or Vol. $ABC = \frac{1}{3} AD \cdot$ (area generated by BC).

Let the student give the proof when the triangle is of other forms, *e.g.* in Case 1 when AD falls *within* the triangle, and in Case 2 when BC is parallel to l.

289. Area of the sphere. About a circle circumscribe a polygon as follows: Construct two diameters AB and CD at right angles to each other and divide each quadrant into an *even* number of parts by points, as D, E, F. At each alternate division point, beginning with the first point D, draw a tangent.



There results a regular polygon with vertices on the diameters AB and CD.

If now we construct another polygon in the same manner by dividing each quadrant into *twice* as many arcs, it will likewise have two vertices on each of the diameters AB and CD, and this may be repeated at pleasure.

Now inscribe a polygon similar to the one just circumscribed by joining the points C and E, E and A, and so on.

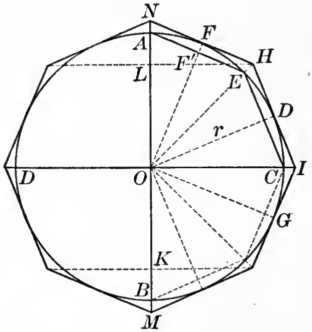
If now the whole figure is made to revolve about AB as an axis, the circle generates a sphere, and the circumscribed and inscribed polygons generate sets of circumscribed and inscribed cones and frustums of cones.

We assume the following:

290. Axiom IX. *A sphere has a definite area and incloses a definite volume which are less respectively than the volume and the surface of any circumscribed figure and greater than those of any inscribed convex figure.*

291. THEOREM. *The area of a sphere is* $4\pi r^2$.

Proof: Denote by r the radius of the circle which generates the sphere. That is, in the figure, $r = OF = OD = OG$.



Then by § 286 the lateral areas of the frustums whose axes are OL and OK are $2\pi r \cdot OL$ and $2\pi r \cdot OK$, and by § 287 the lateral areas of the cones whose axes are LN and KM are respectively $2\pi r \cdot LN$ and $2\pi r \cdot KM$.

Hence, the total surface of the whole circumscribed figure is

$$2\pi r \cdot MK + 2\pi r \cdot KO + 2\pi r \cdot OL + 2\pi r \cdot LN,$$

or $$2\pi r\,(MK + KO + OL + LN) = 2\pi r \cdot MN.$$

If now a polygon of twice the number of sides is constructed, in a manner similar to the above, we obtain another circumscribed figure whose area is $2\pi r \cdot M'N'$ where M' and N' are the two vertices on the line AB.

As this process is repeated indefinitely, the vertices which lie on the line AB may be made to approach as near as we please to A and B, and hence the total surface generated approaches as near as we please to

$$A = 2\pi r \cdot AB = 4\pi r^2.$$

We now prove that A cannot *differ* from $4\pi r^2$.

(*a*) Suppose $A > 4\pi r^2$, and let $A = 4\pi r^2 + d$. (1)

Let k be the difference between $M'N'$ and AB.

Then the area of the circumscribed figure is

$$2\pi r\,(2r + k) = 4\pi r^2 + 2\pi rk.$$

Since k can be made as small as we please, $2\pi rk$ can be made smaller than d.

Hence, by Axiom IX, the area $> 4\pi r^2 + d$, which contradicts the hypothesis (1).

Therefore the area cannot be greater than $4\pi r^2$.

(*b*) In the inscribed figure let OF' be the apothem.

Then by § 286 and an argument like that used above, we find that the area of the inscribed figure is

$$2\pi \cdot AB \cdot OF' = 4\pi r \cdot OF'.$$

By continuing to double the number of sides, OF' may be made to approach as nearly as we please to $OF = r$.

Suppose that $A < 4\pi r^2$ and let $A = 4\pi r^2 - d'$. (2)

Let k' be the difference between r and OF'.

Then the area of the inscribed figure is

$$4\pi r \cdot OF' = 4\pi r(r - k') = 4\pi r^2 - 4\pi rk'.$$

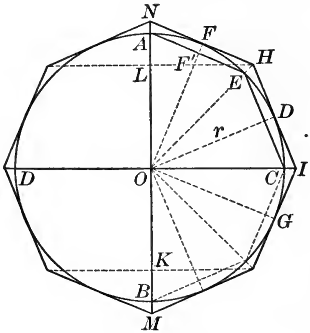
Since k' can be made as small as we please, $4\pi rk'$ can be made less than d'. Hence there is some inscribed figure whose area is greater than $4\pi r^2 - d'$. Hence by Axiom IX, $A > 4\pi r^2 - d'$, which contradicts (2).

Therefore the area cannot be less than $4\pi r^2$. Since the area is neither less nor greater than $4\pi r^2$, it is therefore equal to $4\pi r^2$.

292. THEOREM. *The volume of a sphere is* $\frac{4}{3}\pi r^3$.

Proof: From § 288 the volume of the figures formed by rotating the triangles OIH and OHN about AB as an axis

is $$\frac{r}{3}\cdot(\text{area generated by } IH \text{ and } HN).$$



That is, the volume of the whole circumscribed figure is equal to the surface of the figure multiplied by $\frac{r}{3}$.

Now suppose the volume of the sphere not equal to $\frac{4}{3}\pi r^3$, but equal to $\frac{4}{3}\pi r^3 + d$.

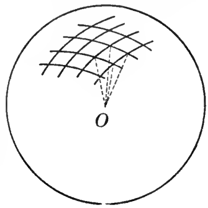
Since by the argument of § 291 we can obtain a circumscribed figure whose surface is as near as we please to $4\pi r^2$, it follows that one can be found whose volume is less than $\frac{4}{3}\pi r^3 + d$.

Hence, the volume cannot be **greater** than $\frac{4}{3}\pi r^3$.

In a similar manner, using the inscribed figures show that the volume cannot be **less** than $\frac{4}{3}\pi r^3$.

293. **The volume of a sphere by inspection.** The fact that the volume of a sphere is equal to its area multiplied by one third its radius may be inferred directly from the accompanying figure.

The sphere is covered with a network of small spherical quadrilaterals. If these are taken small enough, they may be regarded as approximately *plane surfaces.*



On this supposition we have a set of pyramids with a common altitude r and the sum of their bases approximately equal to the area of the sphere.

Hence, their combined volume is $\frac{1}{3} r \times$ (area of sphere) or $\frac{1}{3} r \cdot 4 \pi r^2$. That is, the volume is $\frac{4}{3} \pi r^3$.

It is clear that, by making these quadrilaterals sufficiently small, this result may be approximated as nearly as we please.

294. EXERCISES.

1. The surface of a polyhedron circumscribed about a sphere of radius 4 inches is 420 square inches. Find its volume.

2. The volume of a polyhedron circumscribed about a sphere of radius 3.5 inches is 450 cubic inches. Find its surface.

3. Given a sphere of radius 6 inches, is there any *upper* limit to the volume of its circumscribed polyhedrons? That is, can polyhedrons be circumscribed having a volume as large as we please?

4. On the same sphere is there any *lower* limit to the volume of its circumscribed polyhedrons?

5. Show that the areas of two spheres are in the same ratio as the squares of their radii or of their diameters.

6. Show that the volumes of two spheres are in the same ratio as the cubes of their radii or of their diameters.

7. Find the area of a sphere whose radius is 8 inches.

8. Find the volume of a sphere whose radius is 10 feet.

9. If the area of a sphere is 227 sq. ft., find its radius.

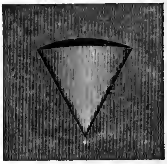
10. If the volume of a sphere is 335 cu. in., find its radius.

11. If the volumes of two spheres are 27 cu. in. and 729 cu. in., compare their radii.

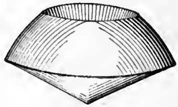
295. Definition. That part of a sphere included between two parallel planes cutting it is called a **zone**. The perpendicular distance between the planes is the **altitude** of the zone.

The figure formed by a zone, together with the circular plane-segments cut out by the sphere, is called a **spherical segment**, and the circular plane-segments are its **bases**.

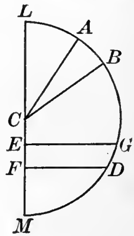
If one of the cutting planes is tangent to the sphere, then the spherical segment and the corresponding zone are said to have but **one base**. The altitude is the perpendicular distance from the base to the tangent plane.



If one nappe of a convex conical surface has its vertex at the center of a sphere, the portion cut off by the sphere, together with the intercepted part of the spherical surface, is called a **spherical cone**.



If two spherical cones have the same axis, one lying within the other, the figure formed by their two lateral surfaces, together with the part of the sphere intercepted between them, is called a **spherical sector**.



If the two cones are right circular cones, they intercept circles on the sphere, and the zone thus included is called the **base** of the spherical sector.

If the accompanying figure be revolved about LM as an axis, then any arc, as GD or MD, generates a zone, the former with two bases, the latter with one.

The figure MDF or $FDGE$ generates a spherical segment, the former with one base, the latter with two.

The figure CAL or CBL generates a spherical cone, CL being the common axis.

The figure CBA generates a spherical sector.

296. Area of a zone. An argument precisely like that of § 291 shows that the area of a zone is

$$A = 2\pi rh$$

where h is the altitude of the zone.

That is, instead of AB, the diameter in case of the sphere, we should have the sum of the altitudes of the frustums circumscribed about the zone equal to h, the altitude of the zone.

297. Volume of a spherical cone. An argument precisely like that of § 292 shows that the volume of a spherical cone is

$$V = \frac{r}{3} \cdot A$$

where A is the area of the zone cut out of the sphere by the cone. Hence, if h is the altitude of this zone, we have

$$V = \frac{r}{3} \cdot 2\pi rh = \frac{2\pi}{3} r^2 h.$$

In like manner the **volume of a spherical sector is**

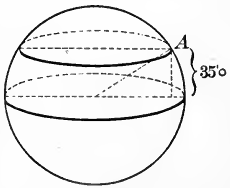
$$V = \frac{2\pi}{3} r^2 h.$$

298. **EXERCISES.**

1. The radius of a sphere is 6 and the altitude of a zone is 5. Find the area of the sphere and of the zone.

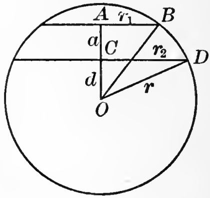
2. The area of a zone is 36π and its altitude 4. Find the radius of the sphere.

3. If the radius of a sphere is r, find the perpendicular distance from a point A to the plane of a given great circle if the distance on the sphere from A to the great circle is 35°.



4. Solve the preceding problem if the distance on the sphere from A to the given great circle is $23\frac{1}{2}$°. Also if the distance is $66\frac{1}{2}$°. (Use the tables, page 57.)

299. PROBLEM. *To find the volume of a spherical segment.*



Solution. Let r be the radius of the sphere, and r_1 and r_2 the radii of the bases of the segment, a the altitude of the zone, and let the segment be generated by revolving the figure $ACDB$ about AO as an axis.

We have Vol. generated by $ODB = \frac{2\pi}{3}r^2a$. § 297

Vol. generated by $OAB = \frac{\pi}{3}r_1^2(a+d)$. (Why?)

Vol. generated by $OCD = \frac{\pi}{3}r_2^2d$. (Why?)

Hence, $V = \frac{2\pi}{3}r^2a + \frac{\pi}{3}r_1^2(a+d) - \frac{\pi}{3}r_2^2d$

$$= \frac{\pi}{3}[2r^2a + r_1^2a + d(r_1^2 - r_2^2)]. \quad (1)$$

From $r^2 = r_2^2 + d^2$ and $r^2 = r_1^2 + (a+d)^2$

we obtain $$d = \frac{r_2^2 - r_1^2 - a^2}{2a}. \quad (2)$$

Substituting this value of d in $r^2 = r_2^2 + d^2$,

we get $$r^2 = \frac{r_2^4 + r_1^4 + a^4 - 2r_1^2r_2^2 + 2a^2r_2^2 + 2a^2r_1^2}{4a^2}. \quad (3)$$

Substituting (2) and (3) in (1) and reducing, we have

$$V = \frac{\pi a}{2}(r_1^2 + r_2^2) + \frac{\pi}{6}a^3 = \frac{a}{2}(\pi r_1^2 + \pi r_2^2) + \frac{4}{3}\pi\left(\frac{a}{2}\right)^3.$$

Hence, we have the result:

THEOREM. *The volume of a spherical segment is numerically equal to the sum of the areas of its bases multiplied by half its altitude, plus the volume of a sphere whose diameter is equal to the altitude of the segment.*

SUMMARY OF CHAPTER V.

1. Collect the theorems involving plane sections of the sphere.

2. Collect the definitions involving plane sections of the sphere.

3. State some of the principal exercises and problems connected with plane sections of the sphere.

4. Collect the definitions on trihedral angles and spherical triangles.

5. Arrange in parallel columns the pairs of theorems on trihedral angles and spherical triangles, which are proved without the use of polar triangles.

6. Collect the definitions on polar triangles.

7. Collect the theorems on polar triangles.

8. Continue the lists started in Ex. 5, adding the theorems proved by means of polar triangles.

9. Make a list of the definitions involving polyhedral angles and spherical polygons.

10. Collect the theorems involving polyhedral angles and spherical polygons.

11. Collect the theorems on the areas of spherical triangles and polygons.

12. Give the definitions and axioms pertaining to the area and volume of the sphere.

13. State all the theorems pertaining to the area and volume of the sphere.

14. Give the definitions and theorems pertaining to spherical figures, such as zones, cones, sectors, segments.

15. Collect all the mensuration formulas in this chapter.

16. Collect all the mensuration formulas of Solid Geometry.

17. Which of these formulas are illustrations of the general theorem that the surfaces of similar solids are in the same ratio as the squares of their corresponding linear dimensions? Which ones are special instances of the theorem that the volumes of similar solids are in the same ratio as the cubes of corresponding linear dimensions?

18. Describe some of the most important applications in this chapter. Return to this question after studying the following list.

PROBLEMS AND APPLICATIONS.

1. What part of the earth's surface lies in the torrid zone? What part in the temperate zones? What part in the frigid zones? The parallels $23\frac{1}{2}°$ north and south of the equator are the boundaries of the torrid zone, and the parallels $66\frac{1}{2}°$ north and south are the boundaries of the frigid zones.

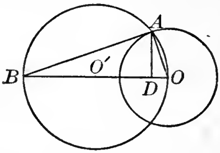
2. Find to four places of decimals the area of a sphere circumscribed about a cube whose edge is 6. No square root is to be approximated in the process, and the value of π is taken 3.1416.

3. Can the volume of the sphere in the preceding exercise be approximated without finding a square root? Find the volume.

4. Find the area of a sphere circumscribed about a rectangular parallelopiped whose sides are a, b, and c.

5. Find the volume of the sphere in the preceding example.

6. A fixed sphere with center O has its center on another sphere with center O'. Show that the area of the part of O' which lies within O is equal to the area of a great circle of the sphere O, provided the radius of the sphere O is not greater than the diameter of O'.



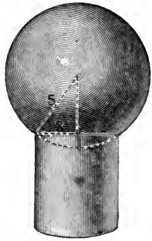
SUGGESTION. Let the figure represent a cross section through the centers of the two spheres. Connect O with A and B. Then $\overline{OA}^2 = OB \times OD$. But OD is the altitude of the zone of O', which lies within O, and OB is the diameter of the sphere O'. Hence, the area of the zone is $\pi\, BO \times OD = \pi\, \overline{OA}^2$.

7. Given a solid sphere of radius 12 inches. A cylindrical hole is bored through it so that the axis of the cylinder passes through the center of the sphere. What area of the sphere is removed if the diameter of the hole is 4 inches?

8. Find the volume removed from the sphere by the process described in the preceding exercise.

9. Through a sphere of radius r a hole of radius r_1 is bored so that the axis of the cylinder cut out passes through the center of the sphere. Find the volume of the sphere which remains.

10. A cylindrical post 6 in. in diameter is surmounted by a part of a sphere 10 in. in diameter, as shown in the figure. Find the surface and the volume of the part of the sphere used.



11. A cylindrical post 5 ft. long and 4 in. in diameter is turned so as to leave on it a part of a sphere 7 in. in diameter and having its center in the axis of the post. Find the volume of the whole post.

12. Find the volume of a spherical shell 1 inch thick if its outer diameter is 8 inches.

13. What is the outer diameter of a spherical shell an inch thick whose volume is half that inclosed within ~~a~~ sphere ~~of the same diameter~~? the inner

14. Compare the volumes and areas of a sphere and the circumscribed cylinder.

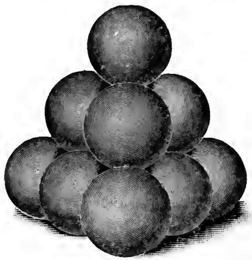
15. In a sphere of radius r inscribe a cylinder whose altitude is equal to its diameter. Compare its volume and area with those of the sphere.

16. In a sphere inscribe a cylinder whose altitude is n times its diameter. Compare its area and volume with those of the sphere.

17. Compute the length of the diagonal of a cube in terms of its side, and also the length of the side in terms of half the diagonal.

18. Express the volume of a cube inscribed in a sphere in terms of the radius of the sphere.

19. Three spheres each of radius r are placed on a plane so that each is tangent to the other two. A fourth sphere of radius r is placed on top of them. Find the distance from the plane to the top of the upper sphere.



20. Find the vertical distance from the floor to the top of a triangular pile of spherical cannon balls, each of radius 5 inches, if there are 3 layers in the pile.

21. Solve a problem like the preceding if there are 16 layers in the pile, each shot of radius r.

22. Solve a problem like the preceding for a square pile of shot of 12 layers.

23. Solve Exercises 21 and 22 if there are n layers in each pile.

CHAPTER VI.

VARIABLE GEOMETRIC MAGNITUDES.

GRAPHIC REPRESENTATION.

300. It is often useful to think of a geometric figure as **continuously varying** in size, or in shape, or both.

E.g. if a parallelopiped has a fixed base, say 24 square inches, but an altitude which varies continuously from 3 inches to 5 inches, then the volume varies continuously from $3 \cdot 24 = 72$ to $5 \cdot 24 = 120$ cubic inches.

We may even think of the altitude as starting at zero inches and increasing continuously, in which case the volume also starts at zero.

From this point of view many theorems may be represented **graphically**. The graph has the advantage of exhibiting the theorem at once for all its particular cases.

For a description of graphic representation, see Chapter V of the authors' High School Algebra, Elementary Course.

301. THEOREM. *The volumes of two parallelopipeds having equal bases are in the same ratio as their altitudes.*

Graphic Representation. By § 86 we have

$$\textit{Volume} = \textit{base} \times \textit{altitude}.$$

Consider parallelopipeds each with a base whose area is A, and with altitudes h_1, h_2, h_3, and corresponding volumes V_1, V_2, V_3.

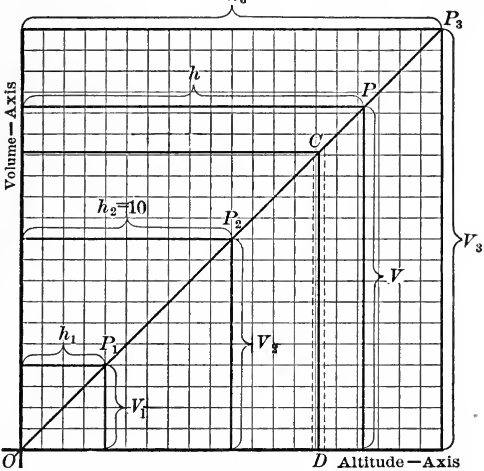
Then $$\frac{V_1}{V_2} = \frac{Ah_1}{Ah_2} = \frac{h_1}{h_2}, \quad \frac{V}{V_3} = \frac{Ah_2}{Ah_3} = \frac{h_2}{h_3}, \text{ etc.} \qquad (1)$$

Consider the case where $A = 10$. Let one horizontal space represent one unit of altitude, and one vertical space ten units of volume.

Thus, the point P_2 has the ordinate $V = 10$ vertical units (representing 100 units of volume), and the abscissa $h_2 = 10$ horizontal units.

Similarly, locate the points P_1 and P_3.

Using equations (1), show that O, P_1, P_2, P_3 lie in a straight line.



If we suppose that while the base of the parallelopiped remains fixed, the altitude varies continuously through all values from $h_2 = 10$ to $h_3 = 20 = 2 \times 10$, then the volume varies continuously from $10 \times 10 = 100$ to $10 \times 20 = 200$.

Using any altitude as an abscissa and the corresponding volume as an ordinate, show, as in § 368, Plane Geometry, that the point so determined lies on the line OP_2P_3.

302. The preceding theorem may also be stated:

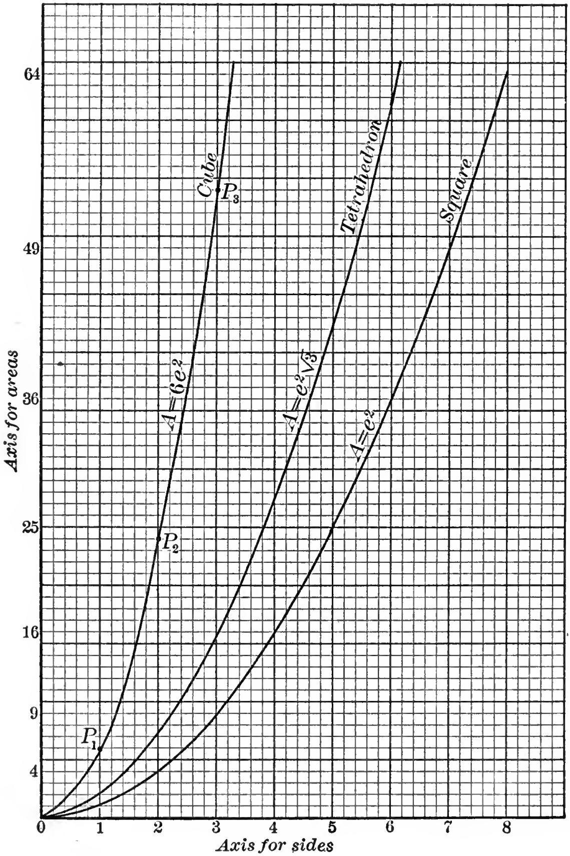
The volume of a parallelopiped with a fixed base varies directly as its altitude.

This means that if V and h are the varying volume and altitude, and V_1 and h_1 the volume and altitude at any given instant, then

$$\frac{V}{V_1} = \frac{h}{h_1} \text{ or } V = \frac{V_1}{h_1} \cdot h, \text{ or } V = kh, \text{ where } k \text{ is the } \textit{fixed ratio } \frac{V_1}{h_1}.$$

The graph representing the relation of two variables when one varies *directly* as the other is a straight line.

303. Problem. *To represent graphically the relation between the area and the edge of a cube as the edge varies continuously.*



Solution. On the horizontal axis lay off segments equal to the various values of the edge e, and on the vertical axis lay off segments equal to the corresponding areas A.

If one vertical space represents one unit of area, then the points P_1, P_2, P_3, etc., lie on the steep curve.

The student should locate many more points between those here shown, and see that a **smooth** curve can be drawn through them all.

304. The graph of the relation between two variables, one of which varies as the square of the other, is always similar to the one just given.

The lowest curve represents the relation between the side of a square and its area, and the third curve represents the relation between the edge of a regular tetrahedron and its total surface.

In each of these cases the area is said to **vary** as the square of the side of the figure. These are special cases of the theorem, § 199, that the surfaces of similar figures are in the same ratio as the squares of their corresponding linear dimensions. This theorem may also be stated:

The areas of similar figures vary as the squares of their linear dimensions.

305. **EXERCISES.**

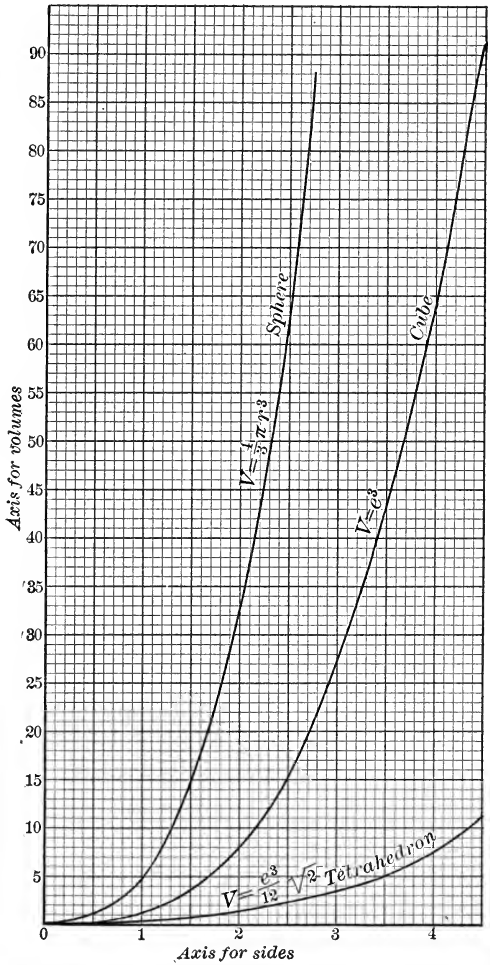
1. From the graph find approximately the area of a regular tetrahedron whose side is 2.5; one whose side is 3.7; 4.3.

2. Find approximately from the graph the edges of cubes whose total areas are 25 square units; 40 square units; 56 square units.

3. Find approximately from the graph the edges of regular tetrahedrons whose total areas are 42 square inches; 17 square inches; 55 square inches.

4. Construct a graph showing the relation between the edge and the total area of a regular octahedron.

306. Problem. *To construct a graph showing the relation between the edge of a cube and its volume.*



Solution. Taking ten horizontal spaces to represent one unit of length of side of the cube and one vertical unit to represent one unit of volume, construct the middle graph shown on the page opposite.

307. The lower graph represents the relation between the length e of the edge of a regular tetrahedron and its volume V; and the upper graph represents the relation between the radius and the volume of a sphere.

In each of these cases the volume is said to vary as the cube of the given linear dimension.

These are special cases of the theorem, § 199, that the volumes of similar solids are in the same ratio as the cube of their ratio of similitude.

This theorem may now be stated:

The volumes of similar solids vary as the cubes of their linear dimensions.

308. EXERCISES.

1. From the graph read off approximately the cubes of 1.3; 2.4; 3.7.

2. From the same graph read off approximately the cube roots of 17; 46; 54; 86.

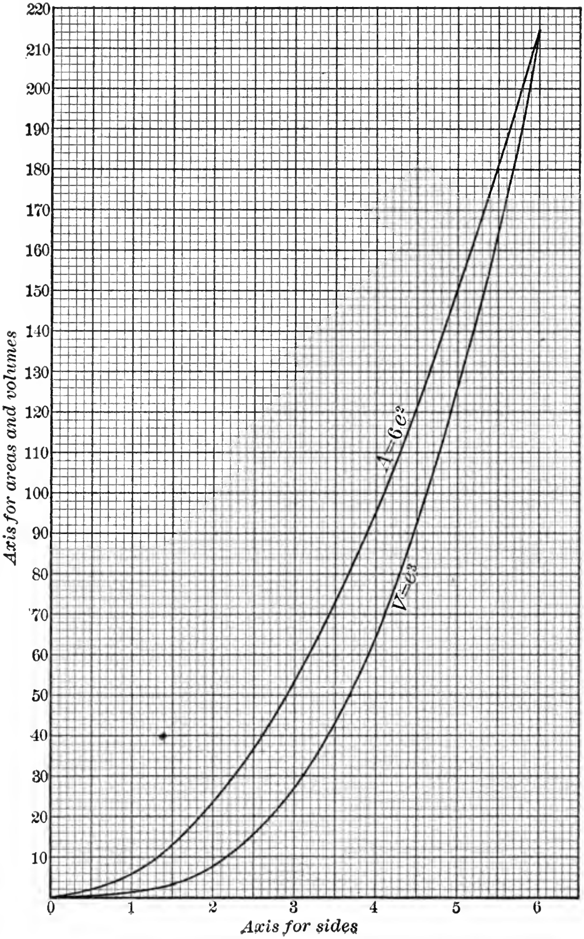
3. Find approximately by means of the graph the volume of a sphere whose radius is 1.5; 2.3; 2.7.

4. What is the radius of a sphere whose volume is 26; 52; 71; 80?

5. Construct a graph giving the relation between the volume and the length of a side of regular octahedrons.

6. By means of the graph just constructed find the volume of a regular octahedron whose side is 1.3; 2.7; 3.6.

7. From the graph constructed under Ex. 5 read off approximately the lengths of an edge of a regular octahedron whose volume is 18; also of one whose volume is 46.


220
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20
10
0
1
2
3
4
5
6
Axis for areas and volumes
Axis for sides
A=6e²
V=e³

309. The graph on the opposite page exhibits the variation of the area and the volume of the cube. We notice that for small values of e the numerical value of the area is greater than that of the volume. For $e = 6$ these values are equal, and for e greater than 6, the numerical value of the volume is greater than that of the area.

This is an instance of the general fact that if a solid figure increases in size but remains similar to its first form, then after a certain point its volume increases more rapidly than its area or the area of any cross section.

310. The fact that one variable y varies as the square or the cube of another variable x is expressed algebraically by the equations $y = kx^2$ and $y = kx^3$, respectively, where k is a constant number to be determined in any particular case.

If the value of k is known, the relations represented by these equations may be shown by a graph like those on the preceding pages.

The equation $y = kx^3$ may be plotted by multiplying the ordinate of each point of the graph $V = e^3$ (see page opposite) by the value of k.

SUMMARY OF CHAPTER VI.

1. Give a brief summary of the graphic representation process in general and in particular, as applied to geometrical figures.

2. Make a list of the theorems which are represented graphically in this chapter.

3. Show in what respects such a representation has advantages over the algebraic representation.

4. State as formulas the important laws of variation for geometric magnitudes discussed in this chapter.

5. Make a collection of important applications given in this chapter, including those which occur in the following set.

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PROBLEMS AND APPLICATIONS.

1. Assuming that the weights of schoolboys vary as the cubes of their heights, construct a graph representing the relation between their heights and weights, if a boy 5 feet 8 inches tall weighs 130 pounds.

SUGGESTION. If w represents the number of pounds in weight and h the number of feet in height, $w = kh^3$. From $w = 130$, when $h = 5\frac{2}{3}$, we have $k =$ ~~.714.~~ .684

For the purpose of the graph, $k = .7$ is accurate enough. Hence, we obtain the required graph by multiplying each ordinate of the graph of $V = e^3$ by .7.

2. From the graph constructed in the preceding example find the weight of a boy 5 feet tall; one 5 feet 4 inches; one 5 feet 6 inches. Compare with the weights of boys in your class.

3. If a man 6 feet tall weighs 185 pounds, construct a graph representing the weights of men of similar build and of various different heights.

4. If steamships are of the same shape, their tonnages vary as the cubes of their lengths. The *Mauretania* is 790 feet long, with a net tonnage of 32,500. Construct a graph representing the tonnage of ships of the same shape, and of various different lengths.

Other ships which at one time or another have held ocean records are: the *Deutschland*, length 686 ft. and tonnage 16,500; the *Kaiser Wilhelm Der Grosse*, length 648 ft. and tonnage 14,300; the *Lucania*, length 625 ft. and tonnage 13,000 (nearly); and the *Etruria*, length 520 ft. and tonnage 8000. By means of this graph decide whether or not these boats have greater or less tonnage than the *Mauretania* as compared with their lengths.

NOTE. The following more general problems further illustrate the wide range of application of the fundamental theorem that the surfaces and the volumes of similar solids vary respectively as the squares and the cubes of their corresponding linear dimensions. These problems need not be solved by means of graphs.

5. Raindrops as they start to fall are extremely small. In the course of their descent a great many are united to form larger and larger drops. If 1000 such drops unite into one, what is the ratio of the surface of the large drop to the sum of the surfaces of the small drops?

6. Consider two machines similar in shape and of heights h_1 and h_2. Since the tensile strength of corresponding parts varies as the areas of their cross sections, it follows that the tensile strengths of corresponding parts are in the ratio $h_1^2 : h_2^2$. But the total volumes of the machines vary as the cubes of the heights; that is, the total weights are in the ratio $h_1^3 : h_2^3$. Does this fact offer any hindrance to the building of machines indefinitely large?

7. The strength of a muscle varies as its cross-section area, which in turn varies as the square of the height or length of an animal, while the weight of the animal varies as the cube of its height or length. Use these facts to explain the greater agility of small animals. For example, compare the rabbit and the elephant.

8. Assuming the velocities the same, the amounts of water flowing through pipes vary directly as their cross-section areas. How many pipes, each 4 in. in diameter, will carry as much water as one pipe 72 in. in diameter?

9. What must be the diameter of a cylindrical conduit which will carry enough water to supply ten circular intakes each eight feet in diameter?

10. A water reservoir, including its feed pipes, is replaced by another, each of whose linear dimensions is twice the corresponding dimension of the first. If the velocity of the water in the feed pipes of the new system is the same as that in the old, will it take more or less time to fill the new reservoir than it did the old? What is the ratio of the new time to the old?

11. If two engine plants are exactly similar in shape, but each linear dimension in one is three times the corresponding dimension of the other, and if the steam in the feed pipes flows with the same velocity in both, compare the speeds of the engines.

12. If two men, one 5 ft. 6 in. and the other 6 ft. 2 in. in height, are similar in structure in every respect, how much faster must the blood flow in the larger person in order that the body tissues of both shall be supplied equally well?

Suggestion. Note that the amount of tissue to be supplied varies as the cube of the height, while the cross-section area of the arteries varies as the square of the height.

CHAPTER VII.

THEORY OF LIMITS.

GENERAL PRINCIPLES.

311. In the Plane Geometry it was found that there are segments which have no common unit of measure, that is, which are **incommensurable**, and that the ratio of the lengths of two such segments could be expressed only approximately by means of integers and ordinary fractions. Other incommensurables occurred in dealing with the length of the circle and the area inclosed by it.

In Chapters III, IV, and V we confined ourselves to an informal first treatment of these incommensurable ratios, tacitly assuming their existence and **computing them approximately**. In Chapter VII their existence was explicitly assumed, and certain theorems proved rigorously on the basis of these assumptions. In the Solid Geometry the areas and volumes of the cylinder, cone, and sphere have been treated according to this latter method.

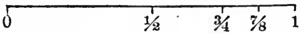
312. In returning to this subject once more we fix our attention on the **incommensurable ratios themselves**, and the method of determining them, rather than on the **process of approach** and the practical computation based on it. We have already used symbols such as $\sqrt{2}$ to represent the ratio of the lengths of incommensurable segments.

In general, the ratio between any two incommensurable geometric magnitudes of the same kind may be represented by what is called an **irrational number**; that is, a number which is *neither an integer nor a quotient of two integers*.

313. The following method for determining irrational numbers is, for simplicity, applied first to the integer 1.

Throughout this discussion the words "point on a line" and "number" will be used interchangeably.

In a straight line mark a certain point 0 (zero), and one unit to the right of it mark another point 1. Also lay off points such that their distances from 0 are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, $\cdots$.



If this sequence of points is carried ever so far, it will never reach the point 1. If, however, we select a point k to the left of 1, no matter how near it, we may always go far enough along this sequence to reach points between k and 1.

314. The point 1 has two definite relations to this sequence.

(*a*) *Every point of the sequence is to the left of* 1.

(*b*) *For any fixed point* K *to the left of* 1 *there are points of the sequence between* K *and* 1.

315. We now note that 1 is the *only* point on the whole line such that both (*a*) and (*b*) are true of it. For every point to the right of 1 (*a*) is true, but (*b*) is not. For every point to the left of 1 (*b*) is true, but (*a*) is not.

It follows therefore that, while the points of the sequence merely *approach* the point 1, the sequence, *taken as a whole, serves to determine that point* just as definitely as if the numeral 1 itself were used to indicate the point.

316. The sequence 3, $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{8}$, $\cdots$ is a **decreasing sequence,** and the number 2 sustains relations to it similar to those described above. That is,

(*a′*) *Every point of the sequence is to the right of* 2.

(*b′*) *For every point* K *to the right of* 2 *there are points of the sequence between* K *and* 2.

317. Definitions. An endless sequence of the sort just described is called an **infinite sequence.**

Not every infinite sequence serves to single out a definite point in the manner shown above. Thus the sequence 1, 2, 3, 4, ... fails to do so, because its terms grow large beyond all bound. Such sequences are said to be **unbounded,** while the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ is **bounded.**

Again, the sequence 1, 2, 1, 2, 1, ... fails to single out a definite point. This sequence is said to be **oscillating,** since its terms increase, then decrease, then increase, etc., while $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ is **non-oscillating.**

318. The number 1 is said to be the **least upper bound** of the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$. That is, 1 is the *smallest number* beyond which the sequence does not go. 1 is also said to be the **limit of the sequence.**

Similarly, 2 is the **greatest lower bound** or the **limit** of the sequence $2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \ldots$; that is, 2 is the *greatest number* such that the sequence contains no number less than it.

319. Axiom X. *Every bounded increasing sequence has a least upper bound, and every bounded decreasing sequence has a greatest lower bound.*

This axiom may also be stated:

Every bounded increasing or decreasing sequence has a limit.

This axiom simply means that every such sequence singles out a definite number in the manner stated.

Thus, if we attempt to approximate the square root of 2, we obtain a sequence 1, 1.4, 1.41, 1.414, 1.4142 ..., having for its limit a *definite number* represented by $\sqrt{2}$, which corresponds to the length of the diagonal of a square whose side is unity.

320. If two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are equal term by term, that is, if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3, \ldots$ then they are one and the same sequence and hence by Ax. X they determine the same number.

The same number may be defined by two different sequences.

Thus each of the sequences, $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ and $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \ldots$, has 1 as its limit.

We notice, however, that no matter what definite number we select in either of these sequences, there is a number in the other greater than it.

321. THEOREM. *Two increasing bounded sequences define the same number as their limit if neither sequence contains a number greater than every number of the other.*

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ denote two infinite sequences, with limits A and B, such that there is no a greater than every b, and no b greater than every a.

To prove that $A = B$.

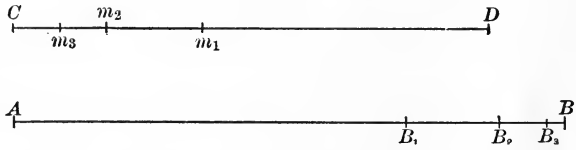
Proof: Suppose that A is not equal to B and that A is less than B. Then there must be numbers of the sequence $b_1, b_2, b_3, \cdots$ greater than A, since by § 314 there are numbers of $b_1, b_2, b_3, \cdots$ greater than any fixed number whatever which is less than B, which contradicts the hypothesis of the theorem.

In like manner we show that B is not less than A. Hence $A = B$.

322. THEOREM. *Two decreasing sequences define the same number as their limit if neither sequence contains a number less than every number of the other.*

APPLICATION OF LIMITS TO GEOMETRY.

323. PROBLEM. *On a given segment* AB *to lay off a sequence of points* B_1, B_2, B_3, *of which* B *is the limit, such that each of the segments* AB_1, AB_2, AB_3, $\cdots$ *is commensurable with a given segment* CD.



Solution. Using m_1, an exact divisor of CD, as a unit of measure, lay off on AB a segment AB_1, such that the remainder B_1B is less than m_1. Then CD and AB_1 are commensurable.

Using a unit m_2, likewise a divisor of CD, and less than B_1B, lay off AB_2 such that B_2B is less than m_2. Then $AB_2 > AB_1$, and CD and AB_2 are commensurable.

Continuing in this manner, using as units of measure segments m_3, m_4, $\cdots$ each an exact divisor of CD and each less than $B_3 B$, $B_4 B$, $\cdots$ respectively, we obtain a sequence of segments AB_1, AB_2, AB_3, $\cdots$, each greater than the preceding and each commensurable with CD.

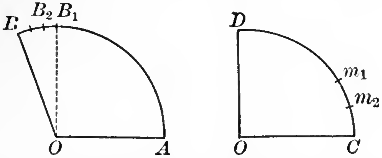
If the units m_1, m_2, m_3, $\cdots$ are so selected that they approach zero as a limit, it follows that B is the limit of the sequence B_1, B_2, B_3, $\cdots$.

If a different sequence of divisors of CD, as m_1', m_2', m_3' $\cdots$, is used, we obtain a sequence B_1', B_2', B_3' $\cdots$, likewise satisfying the conditions of the problem.

We note in regard to any two such sequences B_1, B_2, B_3, $\cdots$ and B_1', B_2', B_3', $\cdots$ determined as above, that they are both *increasing* and that each is such that *no* point of it is to the right of *every* point of the other.

Hence, by § 321 any two such sequences have the same limit B.

324. If two arcs, AB and CD, of the same circle or of equal circles, are given, then, in the same manner as above, points B_1, B_2, B_3, $\cdots$ may be constructed on arc AB, forming a sequence whose limit is B, such that the arc CD is commensurable with every arc of the sequence AB_1, AB_2, AB_3, $\cdots$.



325. Definitions. If B_1, B_2, B_3, $\cdots$ is a sequence of points on the segment AB having the limit B, then the segment AB is said to **be the limit of the segments** AB_1, AB_2, AB_3, $\cdots$.

326. The **ratio of two commensurable segments** has been defined as the quotient of their numerical lengths.

The **ratio of two incommensurable segments** has not been explicitly defined, but it is now possible to do so in terms of the **limit of a sequence.**

Consider two incommensurable segments AB and CD. Let a_1, a_2, a_3, $\cdots$ be the lengths of the segments each commensurable with CD, forming a sequence whose limit is the segment AB, and let b be the length of the segment CD.

Then $\frac{a_1}{b}$, $\frac{a_2}{b}$, $\frac{a_3}{b}$, $\cdots$ is an increasing bounded sequence having a limit which we call R.

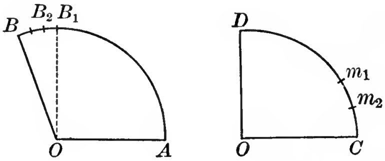
If a_1', a_2', a_3', $\cdots$ are the lengths of another sequence of segments whose limit is AB, the sequence $\frac{a_1'}{b}$, $\frac{a_2'}{b}$, $\frac{a_3'}{b}$, $\cdots$ is another increasing bounded sequence with limit R'.

By § 321 we now know that $R = R'$.

This number R is defined as **the ratio of the segments** AB and CD.

327. The application of the theory of limits to geometry consists chiefly in showing that two numbers are equal because they are the limit of the same sequence, or of sequences having the property stated in the theorem of § 321. In the following paragraphs the applications are made to the chief cases both in plane and in solid geometry.

328. THEOREM. *In the same circle or in equal circles the ratio of two central angles is the same as the ratio of their intercepted arcs.*



Proof: In case the arcs are commensurable the proof is obvious. See § 413, Plane Geometry.

If the arcs AB and CD are not commensurable, let AB_1, AB_2, AB_3, $\cdots$ be a sequence of arcs whose limit is AB, each arc being commensurable with the arc CD.

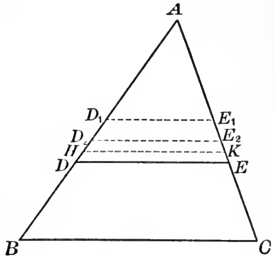
Then the sequence $\frac{AB_1}{CD}, \frac{AB_2}{CD}, \frac{AB_3}{CD}, \cdots$ has a limit R which, by definition, is the ratio of the arcs AB and CD.

Similarly the sequence $\frac{\angle AOB_1}{\angle COD}, \frac{\angle AOB_2}{\angle COD}, \frac{\angle AOB_3}{\angle COD}, \cdots$ has a limit R' which, by definition, is the ratio of $\angle AOB$ and $\angle COD$.

Since $$\frac{AB_1}{CD} = \frac{\angle AOB_1}{\angle COD}, \quad \frac{AB_2}{CD} = \frac{\angle AOB_2}{\angle COD}, \cdots$$

these two sequences are identical and hence define the same limit. Therefore it follows that $R = R'$.

329. Theorem. *A line parallel to the base of a triangle, and meeting the other two sides, divides them in the same ratio.*



Given the $\triangle ABC$ with $DE \parallel BC$ and cutting AB and AC.

To prove that $\dfrac{AD}{AB} = \dfrac{AE}{AC}$.

Proof: Consider the case when AD and AB are incommensurable.

Let $D_1, D_2, D_3 \cdots$ be a sequence on AB whose limit is D. Through these points draw parallels to BC, meeting AC in $E_1, E_2, E_3, \cdots$.

Then E is the limit of the sequence $E_1, E_2, E_3, \cdots$.

For suppose it is not, and that there is a point K on AE such that there is no point of $E_1, E_2, E_3, \cdots$ between K and E. Then draw a line parallel to BC through K, meeting AB in H.

But there are points of the sequence D_1, D_2, D_3, *between* H and D, and hence points of the sequence $E_1, E_2, E_3, \cdots$ *between* K and E, which shows that E is the limit of $E_1, E_2, E_3, \cdots$

Now $\dfrac{AD_1}{AB} = \dfrac{AE_1}{AC}, \dfrac{AD_2}{AB} = \dfrac{AE_2}{AC}, \dfrac{AD_3}{AB} = \dfrac{AE_3}{AC}, \cdots$

Hence, the two sequences,

$$\frac{AD_1}{AB}, \frac{AD_2}{AB}, \frac{AD_3}{AB}, \cdots \text{ and } \frac{AE_1}{AC}, \frac{AE_2}{AC}, \frac{AE_3}{AC}, \cdots,$$

are identical and define the same limit;

that is, $$\frac{AD}{AB} = \frac{AE}{AC}.$$

330. Definitions. If $a_1, a_2, a_3, \cdots$ is a sequence with limit a, then $ka_1, ka_2, ka_3, \cdots$ is a sequence with limit ka.

If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two sequences with limits a and b respectively, then ab is the limit of the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$.

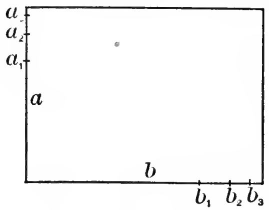
Similarly if $a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots$ and $c_1, c_2, c_3, \cdots$ are sequences with limits a, b, c, then abc is the limit of the sequence $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots$.

For a complete treatment it would be necessary to show that these definitions of multiplication of irrational numbers are consistent with the rest of arithmetic and also that these new sequences are such as to determine definite limits. That, however, is beyond the scope of this book.

If the sides of a rectangle are incommensurable with the unit segment, we **define its area** as follows:

Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers whose limit is the altitude a, and let $b_1, b_2, b_3, \cdots$ be a sequence whose limit is the base b.

Then the **area of the rectangle is the limit of the sequence** $a_1b_1, a_2b_2, a_3b_3, \cdots$.



But by definition the limit of $a_1b_1, a_2b_2, a_3b_3, \cdots$ is the product ab. Hence we have the

331. THEOREM. *The area of a rectangle is the product of its base and altitude.*

332. Definition. In a circle inscribe a sequence $P_1, P_2, P_3, \cdots$ of regular polygons, each having twice the number of sides of the one preceding it.

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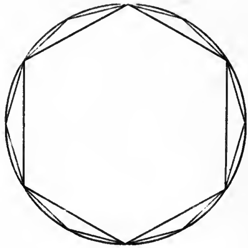
Let the perimeters of these polygons be $p_1, p_2, p_3, \cdots$ and their areas $A_1, A_2, A_3, \cdots$. Then the length c of the circle is the limit of the sequence $p_1, p_2, p_3, \cdots$ and its area A is the limit of $A_1, A_2, A_3, \cdots$.



The sequence of polygons thus inscribed is called an **approximating** sequence of polygons.

That these sequences are *increasing* and *bounded* is obvious from the figure.

333. THEOREM. *The lengths of two circles are in the same ratio as their radii, and their areas are in the same ratio as the squares of their radii.*

Proof: Let the radii of the circles whose centers are O and O' be r and r'. Denote $\frac{r'}{r}$ by k. Then $r' = kr$.

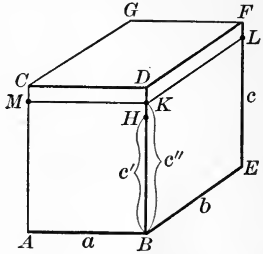
Inscribe in O an approximating sequence of polygons with perimeters $p_1, p_2, p_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$. In O' inscribe a sequence of similar polygons. By §§ 347, 348, Plane Geometry, the perimeters of the latter are $kp_1, kp_2, kp_3, \cdots$ and their areas are $k^2A_1, k^2A_2, k^2A_3, \cdots$.

By § 330, if the limit of $p_1, p_2, p_3, \cdots$ is c and the limit of $A_1, A_2, A_3, \cdots$ is A, then the limits of $kp_1, kp_2, kp_3, \cdots$, and $k^2A_1, k^2A_2, k^2A_3, \cdots$ are kc and k^2A, respectively.

That is, the ratio of the lengths of the circles is $\frac{kc}{c} = k = \frac{r'}{r}$, and the ratio of their areas is $\frac{k^2A}{A} = k^2 = \frac{r'^2}{r^2}$.

If regular circumscribed polygons are used, we obtain *decreasing* sequences both for the perimeters and the areas. That the limits of these sequences are identical with those obtained above is a direct consequence. See § 417, Plane Geometry.

334. Definition. If the three concurrent edges a, b, c of a rectangular parallelopiped are incommensurable with the unit segment, the volume inclosed is defined as follows:



Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers whose limit is the side a. Let $b_1, b_2, b_3, \cdots$ and $c_1, c_2, c_3, \cdots$, be similar sequences whose limits are respectively the dimensions b and c.

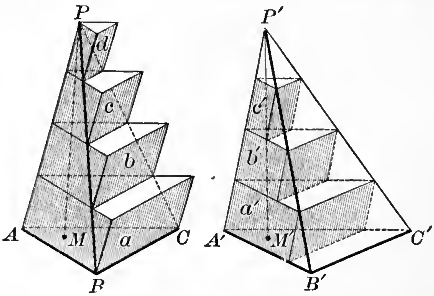
Then the **volume is the limit of the sequences** $a_1b_1c_1, a_2b_2c_2, a_3b_3c_3, \cdots$.

But the limit of this sequence is by definition the product of the limits of the three sequences $a_1, a_2, a_3, \cdots$, $b_1, b_2, b_3, \cdots$, $c_1, c_2, c_3, \cdots$, or abc.

Hence, we have the

335. THEOREM. *The volume inclosed by a rectangular parallelopiped is equal to the product of its three concurrent edges.*

336. Definition. In a triangular pyramid inscribe and circumscribe prisms as shown in the figure. See § 136 for description of the construction.



Denote the sum of the volumes of the inscribed prisms by V_1.

Using as altitudes one half the altitudes of the first set of prisms, inscribe a second set the sum of whose volumes is V_2. Continuing in this manner, we obtain a sequence of sets of prisms with volumes $V_1, V_2, V_3, \cdots$. The limit V of this sequence we **define as the volume of the pyramid.**

If circumscribed prisms are used, we get a *decreasing* sequence of volumes V_1', V_2', V_3', $\cdots$ with limit V'. That these two limits are identical follows from the fact that the sum of the volumes of the circumscribed prisms exceeds that of the inscribed prisms by exactly the volume of the lowest circumscribed prism (see § 139), and this may be made as small as we please.

337. EXERCISE.

1. Prove that the definition of § 334 gives the same volume for the rectangular parallelopiped, no matter what sequences of segments are used, provided their limits are the segments a, b, c.

A proof that the definition of § 336 gives the same volume for the triangular pyramid no matter into how many equal parts the altitude is first divided is rather too complicated to be attempted here. It may be proved by showing that if the altitude is divided into n and also into m equal parts, then if $n > m$, the set of $n - 1$ prisms will be greater than the set of $m - 1$ prisms.

338. THEOREM. *Two pyramids with the same altitudes and equal bases inclose equal volumes.*

Proof: Divide the two equal altitudes into the same number of equal parts and construct inscribed prisms as in § 136.

Since corresponding sets of prisms have equal volumes, it follows that the volumes of the pyramids are the limits of the same sequence, and hence identical.

339. Convex curves. Without specifically defining a **convex closed curve** (see § 97), we assume that in such a curve it is possible to inscribe a sequence of polygons P_1, P_2, P_3, $\cdots$, having perimeters p_1, p_2, p_3, $\cdots$ and areas A_1, A_2, A_3, $\cdots$ with limits p and A respectively, and to circumscribe a sequence of polygons P_1', P_2', P_3', $\cdots$, having perimeters p_1', p_2', p_3', $\cdots$ and areas A_1', A_2', A_3', $\cdots$ with limits p' and A' respectively, such that $p = p'$ and $A = A'$.

These limits p and A we now define **as the perimeter and the area respectively of the curve.**

340. Definition. Given any cylinder with a convex right cross section and an element e. In this cross section inscribe a sequence P_1, P_2, P_3, $\cdots$ of polygons, as in § 332, with perimeters p_1, p_2, p_3, $\cdots$ and areas A_1, A_2, A_3, $\cdots$, thus defining the perimeter p and the area A of the cross section.

Consider a set of prisms inscribed in this cylinder, of which P_1, P_2, P_3, $\cdots$ are right cross sections.

Then the areas and the volumes of these prisms are respectively p_1e, p_2e, p_3e, $\cdots$ and A_1e, A_2e, A_3e, $\cdots$.

The **lateral area and the volume of the cylinder are now defined as the limits of these sequences.** But these limits are by § 330 equal to pe and Ae respectively.

Hence, we have the

341. THEOREM. *The lateral area of a cylinder is the product of an element, and the perimeter of a right section and its volume is the product of an element and the area of a right section.*

342. EXERCISES.

1. Give examples other than those given in the text of infinite sequences which do not determine definite numbers.

2. Give two distinct sequences which determine the number 2. Show that the theorem of § 321 applies and proves that these sequences determine the same number.

3. Give two decreasing sequences each of which determines the number 3. Apply § 322 to show that these sequences determine the same number.

4. State fully the relation between a bounded increasing sequence and the number determined by it. State also the relations between a bounded decreasing sequence and the number determined by it.

5. State fully what is meant by "a limit of a sequence" both for increasing and decreasing sequences.

6. Given two incommensurable segments AB and CD. Lay off on the line AB a decreasing sequence of segments, each of which is commensurable with CD, such that the limit of the sequence is the segment AB.

7. If $a_1, a_2, a_3, \cdots$ is an increasing sequence defining the number 4, prove that $3\,a_1, 3\,a_2, 3\,a_3, \cdots$ defines the number $3 \times 4 = 12$.

Note that in case the sequence $a_1, a_2, a_3, \cdots$ defines an irrational number, we should not be able to prove the corresponding proposition without first defining what is meant by the product of a rational and an irrational number. But such definition would in that case be the proposition itself. See § 330.

8. If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are increasing sequences defining the numbers 3 and 5, show that the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$ defines the number 15.

In case these sequences defined irrational numbers, could a corresponding proposition be proved?

9. In the same manner as in § 326 define the ratio of two incommensurable arcs.

10. Show as above that the ratio so obtained is independent of the sequence of units of measurement used, so long as the limit of this sequence is zero.

11. Treat the ratio of two incommensurable angles in a manner similar to the treatment of arcs in the two preceding exercises.

12. Define the lateral area of a cylinder by means of circumscribed prisms, and show that this definition leads to the same result as that given in § 340.

13. Prove as above that the volume of any convex cylinder is equal to the product of its altitude and area of its base.

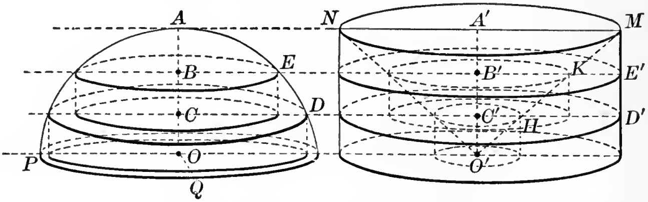
14. Prove that the lateral area of a right circular cone is equal to half the product of the slant height and the perimeter of its base.

15. Prove that the volume of any convex cone is equal to one third the product of its altitude and the area of its base.

Suggestion. The treatment required in the last three exercises is a very close paraphrase of the definitions and proof given in § 334. Observe that we cannot begin to make a proof until we have *defined* the subject matter of the theorem. That is, we must first **define the areas and volumes in question.**

VOLUME OF THE SPHERE.

343. Through two points P and Q of a sphere pass a great circle forming a hemisphere with center O.



Divide OA, the radius perpendicular to the plane of the great circle, into the equal parts OC, CB, and BA. Through C and B pass planes parallel to the plane of POQ, meeting the circle in points D and E, respectively.

Construct right circular cylinders with axes OC and CB and radii CD and BE. Denote by V_1 the sum of the volumes of these cylinders.

Now divide the radius OA into six equal parts and construct five cylinders in the same manner as above. Let the sum of these volumes be V_2.

Continuing in this manner, each time dividing OA into twice as many equal parts as in the preceding, we obtain a sequence of sets of cylinders and a corresponding sequence V_1, V_2, V_3, $\cdots$ of volume.

We now define **the volume inclosed by the hemisphere as the limit of the sequence** V_1, V_2, V_3, $\cdots$.

344. Construct a right circular cylinder with its base in the plane of POQ and altitude $O'A' = OA$.

Denote by F the figure formed by the lower base of the cylinder, its lateral surface and the lateral surface of the cone whose base is the upper base of the cylinder and whose vertex is at O'.

Draw segments $O'M$ and $O'N$. Let the planes through C and B cut $O'A'$ in C' and B' and $O'M$ in H and K.

Now form the cylinder $O'C'H$ whose axis is $O'C'$ and whose radius is $C'H$. Likewise form $C'B'K$.

Let V_1' denote the sum of the volumes of $O'C'D'$ and $C'B'E'$ minus the sum of the volumes $O'C'H$ and $C'B'K$.

In a similar manner, using the planes which divide OA and hence $O'A'$ into six equal parts, we form another set of five cylinders, the sum of whose volumes minus that of the smaller inside cylinders we denote by V_2'.

Continuing in this manner, we obtain a sequence of volumes V_1', V_2', V_3', $\cdots$ whose limit V' we **define as the volume of the given figure F.**

We now prove that $V_1 = V_1'$, $V_2 = V_2'$, $\cdots$.

Denote OA by r, and note that $O'B' = B'K$.

(1) Vol. $C'B'E'$ − Vol. $C'B'K = \pi C'B'(\overline{B'E'}^2 - \overline{B'K}^2) = \pi C'B'(r^2 - \overline{O'B'}^2)$.

(2) Vol. $CBE = \pi CB \cdot \overline{BE}^2 = \pi CB(r^2 - \overline{OB}^2)$, since $\overline{BE}^2 = r^2 - \overline{OB}^2$.

But $OB = O'B'$ and $CB = C'B'$.

Hence, Vol. CBE = Vol. $C'B'E'$ − Vol. $C'B'K$.

Similarly we show that

$$\text{Vol. } OCD = \text{Vol. } O'C'D' - \text{Vol. } O'C'H.$$

Hence, $V_1 = V_1'$. In like manner $V_2 = V_2'$, $V_3 = V_3'$, $\cdots$.

Hence, $V = V'$, since they are the limits of the same sequences.

But the volume of the cylinder $O'A'M$ is πr^3 and of the cone whose volume was subtracted, $\frac{\pi}{3} r^3$. That is, the volume of F is $\frac{2\pi}{3} r^3$, and hence that of the hemisphere is $\frac{2\pi}{3} r^3$.

Hence, we have the

345. THEOREM. *The volume of the sphere is* $\frac{4}{3}\pi r^3$.

346. Note that the above proof consists essentially in showing that the area of the circle BE is equal to that of the ring between the circles $B'E'$ and $B'K$, and that the area of the circle CD is equal to that of the ring between $C'D'$ and $C'H$ and so on.

Indeed this theorem and also that of § 139 are special cases of what is known as

Cavalieri's Theorem. *If two solid figures are regarded as resting on the same plane b, and if in every plane parallel to b the sections of the two figures have equal areas, the figures have equal volumes.*

The proof of this more general theorem is more difficult than any thus far given, inasmuch as it involves sequences which *oscillate;* that is, which are neither constantly increasing nor constantly decreasing.

THE AREA OF THE SPHERE.

347. About a sphere of radius r construct a sequence of circumscribed polyhedrons such that the largest face in each polyhedron becomes as small as we please when we proceed along the sequence. Let $s_1, s_2, s_3, \cdots$ be the total surfaces of these polyhedrons. This forms a decreasing sequence with limit S which we define as the **surface of the sphere.**

The volumes of these polyhedrons will be $\frac{r}{3}s_1, \frac{r}{3}s_2, \frac{r}{3}s_3, \cdots$.

Then the volume V of the sphere is defined as the limit of this sequence of volumes.

Hence, by § 330, $V = \frac{r}{3}S$. But by § 245, $V = \frac{4}{3}\pi r^3$.

Then $$S = \frac{3}{r} \cdot \frac{4}{3}\pi r^3 = 4\pi r^2.$$

Hence, we have the

348. THEOREM. *The area of the sphere is* $4\pi r^2$.

This argument is incomplete in that two distinct definitions have been given of the volume of the sphere, namely, as the limit of an increasing sequence of volumes of inscribed cylinders and also as the limit of a decreasing sequence of volume of circumscribed polyhedrons. But it has not been proved that these two definitions are equivalent; that is, that they lead to the same formula for the volume of the sphere.

The treatment of the area and volume of the sphere in most of the current text-books on geometry is open to a similar objection. Usually two distinct definitions are used for the area of the sphere, namely, as a limit of the areas of circumscribed frustums of cones and also as the limit of the surfaces of circumscribed polyhedrons.

The argument used in §§ 285–293 is not subject to any such objection.

SUMMARY OF CHAPTER VII.

1. Explain the separate stages of treatment of incommensurables and of limits as given in this text, including both the Plane and Solid Geometry.

2. Give an outline of the introduction to this final treatment by means of sequences.

3. Make a list of the definitions, principles, and theorems upon which the treatment of sequences is based.

4. Explain fully the way in which the principles of sequences are applied to geometric theorems. Illustrate by line-segments and arcs.

5. Make a list of the theorems which are proved here by the theory of limits, using the sequence process.

6. State in some detail how this process is used in case of the area and volume of the sphere.

7. Give all the mensuration formulas proved in this chapter.

8. Review the entire list of mensuration formulas for both Plane and Solid Geometry.

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